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THE ASTROPHYSICAL JOURNAL

AN INTERNATIONAL REVIEW OF SPECTROSCOPY AND ASTRONOMICAL PHYSICS

EDITED BY

GEORGE E. HALE
Mount Wilson Observatory of the Carnegie Institution of Washington

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VOLUME LIX JUNE 1924 NUMBER 5

POLE-EFFECTS AND PRESSURE-SHIFTS IN THE LINES OF THE SPECTRA OF ZINC AND CALCIUM

By ROSCOE EVERETT HARRIS

ABSTRACT

Historical review.—Commencing with the work of Humphreys on the pressure-shift of the spectral lines in 1895 a brief statement is made with respect to the work of St. John and Ware, Gale and Adams, F. Goos, Swaim, Royds, and others during the past ten years.

Methods and apparatus.—A 21-foot concave grating with Rowland mounting was employed, and the photographs were made in the second-order spectrum where the dispersion was 1.32 A per millimeter. The description of apparatus is accompanied by detailed diagrams.

The pole-effects and pressure-shifts of the series lines of calcium and of calcium chloride.—In all series both types of shift were found to be proportional to a power of the frequency. The pole-effect differed in the two sources, being greater at the positive pole in calcium, and greater at the negative pole in calcium chloride.

The pole-effect in zinc lines from zinc, brass, and zinc chloride.—A difference in the pole-effect in the zinc lines due to different sources, zinc and zinc chloride, similar to the difference for the metal and salt of calcium, was found. The pole-effect at the positive pole of the brass source changed greatly with changes in the current. The shifts due to pole-effect and pressure were similar Shifts of the series lines were found to vary also with the frequency. Tables of the pole-effects for the zinc lines are given.

HISTORICAL SUMMARY

Prior to 1895 Humphreys had investigated the phenomena of pressure-shift of spectral lines for many substances. While experimental evidence still substantiates most of the conclusions summarized from his results, it is well to note a few important exceptions.

1. All lines do not shift toward the longer wave-length with increased pressure, some being symmetrical and others actually

shifting toward the shorter wave-length. This fact was first definitely established by St. John and Ware,[1] and by Gale and Adams,[2] in 1912–1913. In measuring the pressure-shift of iron lines they found five types of pressure-shift. Those classified as "d" have large shifts toward the red. Those of the "e" group have large shifts toward the violet.

2. The shift of similar lines is not proportional to the wave-length, but is usually inversely proportional to the wave-length for a series.

As a first step toward the formulation of a satisfactory theory based upon experimental facts, Swaim,[3] in 1914, worked out the pressure-shifts for the zinc triplet series, hoping this would initiate investigation of series relationships of many substances. He found the shift proportional to some power of the frequency for each particular series.

In 1915, T. Royds[4] pointed out many interesting facts about the series of the alkaline earth group. He mentioned an apparent inconsistency in the diffuse triplet series, the first member of which seems to be shifted toward the red, the second member being nearly symmetrical, and the other members shifting to the violet. He also showed that a relation existed between pressure-shift and pole-effect.

In 1913, F. Goos[5] found that he could obtain changes in wave-length in certain lines of the nature of pressure-shifts by keeping the pressure the same, and comparing the radiation emitted from different parts of the same arc. He also found the amount of the shift to depend upon the current, diameter of the pole, and arc gap.

Following Royds's work on the alkaline earths, Whitney[6] made a careful measurement of the pole-effect in calcium, and Miller[7] took up the problem of pressure-shift for the same lines.

[1] *Astrophysical Journal*, **36**, 14, 1912.

[2] *Ibid.*, **35**, 10, 1912. [3] *Ibid.*, **40**, 137, 1914.

[4] *Kodaikanal Observatory Bulletin*, Nos. 38 and 40; *Astrophysical Journal*, **41**, 154, 1915.

[5] "Wave-lengths of Iron Lines," *Astrophysical Journal*, **38**, 141, 1913.

[6] *Astrophysical Journal*, **44**, 65, 1916.

[7] *Ibid.*, **53**, 224, 1921.

Gale and Whitney[1] reported the peculiar fact that when ordinary carbon rods were charged with calcium by immersing them in solutions of calcium chloride, the intensity of the lines was strongest at the negative pole and diminished to the positive pole, which was opposite the intensity gradient when both poles were metallic calcium. The pole-effect was also found to be opposite. They concluded from their measurements that when light taken from any part of a subordinate series line is compared with that taken from a stronger part of the same line, there is a difference in wave-length similar to that caused by an increase in pressure, the wave-length at the stronger part of the line correponding to the wave-length at the greater pressure.

APPARATUS AND METHOD

In this work pole-effects and pressure-shifts of lines in the spectra of zinc and calcium were measured, arc sources of different kinds and different currents being employed. The problem was suggested by Professor Gale, for whose valued advice throughout the work I am very grateful.

A 21-foot concave grating, Rowland mounting, was employed. Plates were made in the second order where the dispersion was 1.32 A per millimeter. Horizontal arcs connected in a 220-volt D.C. circuit were used. The potential drop across the burning arc ranged from 35 to 50 volts. A number of measurements of each plate was made on a Gaertner comparator.

For the pressure-shift work the vacuum chamber shown in Figure 1 was used. The details of the cooling system, electrode adjustments, etc., are shown in cross-section and in full front view. A cone-shaped face (not shown in the cut) fits on the front of the chamber. This face holds a quartz window far enough from the arc to escape overheating. An opening in the back of the chamber leads to an oil pump. A pressure range of from 76 cm to 5 cm was used.

The complete arrangement of apparatus is shown in Figure 2. In order to eliminate mechanical disturbances, all exposures were made through the same lens. Exposures to be compared were

[1] *Ibid.*, **43**, 161, 1916.

"decked" by means of a triple comparison slot placed at the focus of the grating conjugate to the plate holder for horizontal lines. This comparison slot, shown in a side sketch of Figure 2, was very useful in pole-effect work. Lines from both poles and from the center of the arc could be obtained as direct comparisons, thus

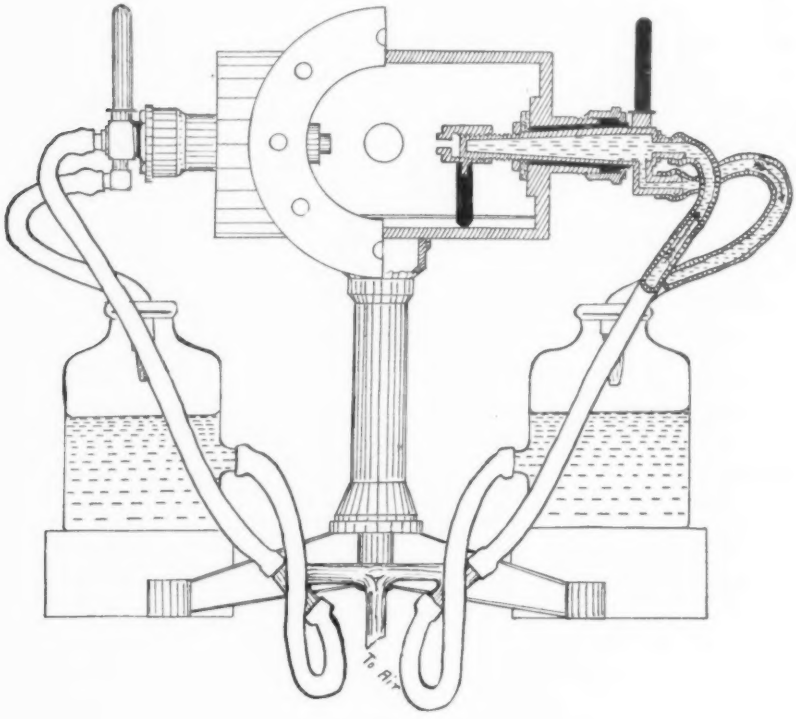


FIG. 1

eliminating much work of cross-comparing. This also avoided mechanical disturbances at the plate holder. The vacuum chamber was supported by a counter-weight to make it easily handled. In use it was mounted on the support in place of the open-air arc. A shutter for the slit, movable from the operator's seat, also proved very valuable. All adjustments, switches, rheostats, and meters were easily accessible from one seat, so that mechanical disturbances

of the operator moving about were avoided. Exposures were made between midnight and six o'clock in the morning.

POLE-EFFECT IN LINES OF ZINC

In this work the pole-effect of the lines of zinc was measured under the same conditions used by Swaim[1] in measuring the pressure-shift of these lines. Electric arcs of metallic zinc electrodes, of

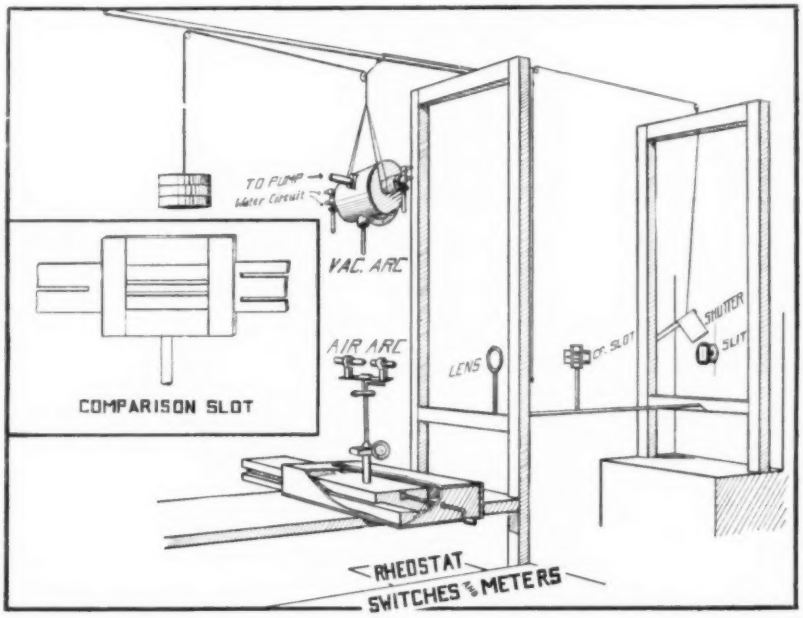


FIG. 2

brass electrodes, and of carbon electrodes filled with zinc chloride were employed. The influence of current changes on the pole-effect was measured. Differences in the character of the lines from the three sources were noted.

Results of measurements of pole-effect are shown in the accompanying table. Series classification, according to Fowler's notation, appears in column 2 of this table. Column 3 is the measured difference in wave-length between the positive-pole radiation and

[1] Swaim used a 2.5 mm arc gap and a 3.5 amp. current.

TABLE I

λ	Series*	Pole-Effect				Swaim's§ Pressure-Effect
		Brass		Zinc		
		+ − c†	− − c‡	+ − c†	− − c‡	
4810.53	1p–1s	0.0020	0.0172	0.0200	−0.0104	0.0099
4722.16	1p–1s	0.0010	0.0170	0.0194	−0.0101	0.0098
4680.20	1p–1s	0.0010	0.0168	0.0162	−0.0101	0.0099
3072.07	1p–2s	0.0020	0.0290	0.0300	−0.0157	0.0163
3035.81	1p–2s	0.0015	0.0288	0.0270	−0.0153	0.0165
3018.38	1p–2s	0.0010	0.0280	0.0200	−0.0153	0.0160
2712.48	1p–3s	0.0030	0.0340	0.0370	−0.0183	0.0204
2684.06	1p–3s	0.0020	0.0340	0.0350	−0.0182	0.0199
2670.44	1p–3s	0.0020	0.0337	0.0310	−0.0182	0.0176
2567.80	1p–4s	.....	.....	.....	.....	0.0224
3345.96	1p–2d	0.0022	0.0100	0.0160	−0.0050	0.0046
3345.51	1p–2d	0.0023	0.0105	0.0170	−0.0055	0.0045
3344.91	1p–2d	0.0025	0.0108	0.0190	−0.0060	0.0047
3302.91	1p–2d	0.0023	0.0100	0.0170	−0.0041	0.0044
3302.56	1p–2d	0.0021	0.0105	0.0180	−0.0056	0.0044
3282.28	1p–2d	0.0022	0.0100	0.0150	−0.0041	0.0045
2801.07	1p–md	.....	.....	.....	.....	0.0082
2800.90	1p–md	0.0040	0.0195	0.0330	−0.0900	0.0088
2770.95	1p–md	0.0035	0.0190	0.0310	−0.0890	0.0085
2770.84	1p–md	.....	.....	.....	.....	0.0077
2756.43	1p–md	0.0032	0.0185	0.0300	−0.0880	0.0092
2608.55	1p–md	0.0051	0.0250	0.0450	−0.0120	0.0112
2582.38	1p–md	0.0044	0.0245	0.0420	−0.0110	0.0091
2569.80	1p–md	0.0045	0.0244	0.0350	−0.0100	0.0085
6362.58	.....	.....	.....	.....	.....	0.0300
4629.88	1P–mD	0.0090	0.0204	.....	.....	0.0120
4058.02	.....	0.0014	0.0080	.....	.....	0.0057
3739.97	unclassif.	0.0020	0.0107	.....	.....	0.0084
3572.90	.....	0.0008	0.0100	.....	.....	0.0085
3075.88	1S–1p_2	0.0001	0.0035	.....	.....	0.0021
2802.70	unclassif.	0.0010	0.0030	.....	.....	0.0240

*Fowler's notation.

† Difference between positive pole and center of arc.

‡ Difference between negative pole and center of arc.

§ *Astrophysical Journal*, **40**, 137, 1914.

the center of the brass arc. Column 4 is the same for the negative pole and the center of the arc. Columns 5 and 6 are the correspond-

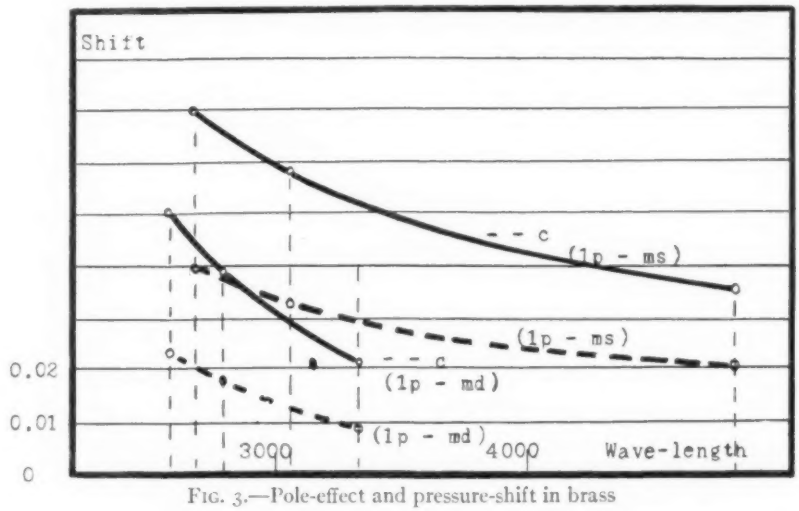


FIG. 3.—Pole-effect and pressure-shift in brass

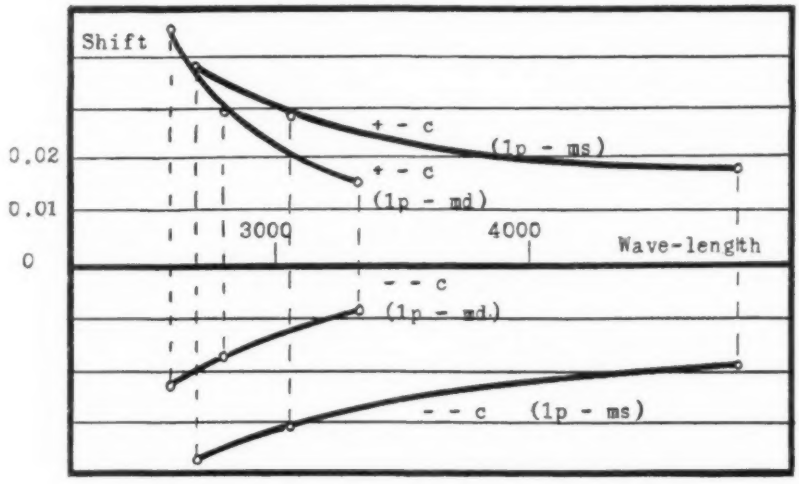


FIG. 4.—Pole-effect from zinc sources

ing measurements for zinc electrodes. The last column contains values of pressure-shift obtained by Swaim.

Measurements of the lines from the zinc chloride source are not given here. The amounts of these shifts were very small. The greatest shift occurred at the negative pole where the line was broadest. The line was very sharp at the positive pole.

In the zinc arc, a line is narrower from the negative pole than from any other part of the source. Since the unsymmetrical broadening is toward the red, the difference in the position of the line at the negative pole and at the center of the arc $[--c]$ is negative, and the values of the pole-effects in column 6 are so

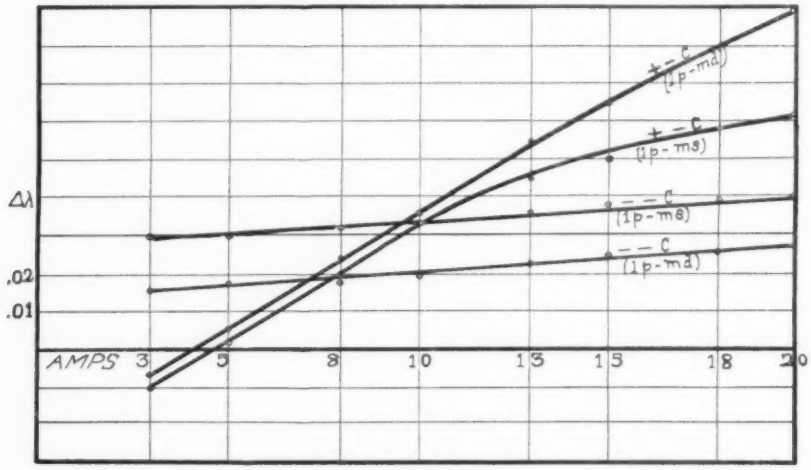


FIG. 5.—Current-effect

marked. On the other hand, a line from the positive pole is very broad, a large pole-effect occurring between the positive pole and the center of the arc $[+-c]$, and even larger between the positive pole and the negative pole $[+--]$. This is the reverse of the case for zinc chloride.

In the brass arc very little shift was found between the positive pole and the center of the arc $[+-c]$ for the conditions here used. As later measurements show, a line from the positive pole of a brass arc widens rapidly with increasing current. For that reason, the shift at the negative pole was used for comparison with pressure-shifts.

This comparison with Swaim's pressure-shift measurements is given in Figure 3 for the triplet series ($1p-md$) and ($1p-ms$). The pressure-shifts are represented by dotted lines. Comparing Figures 3 and 4, it is evident that the pole-effects for zinc (Fig. 4)

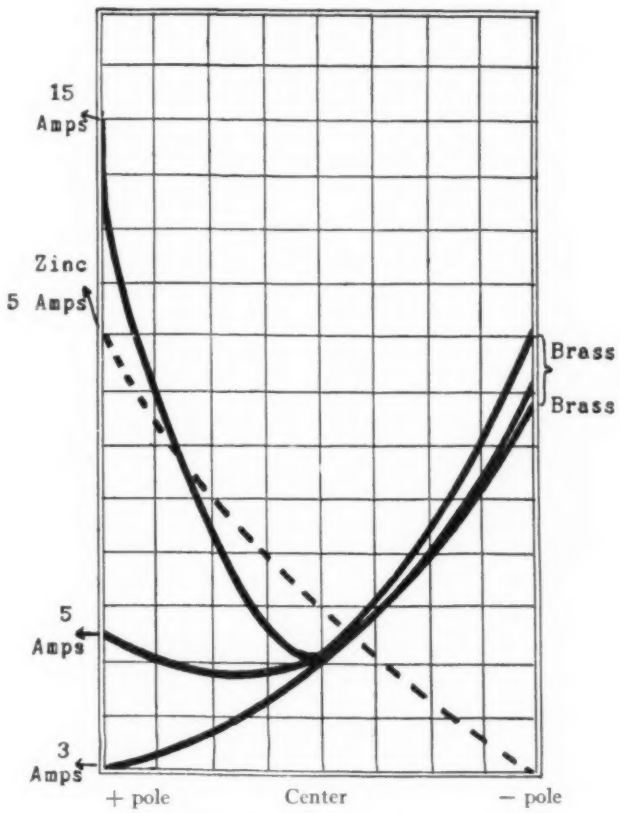
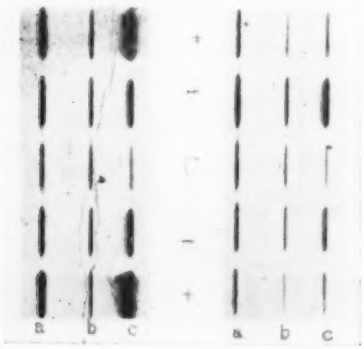


FIG. 6.—Comparison of pole-effect of zinc and of brass

vary according to about the same laws as do those of the same lines from the brass source. The pole-effect $[--c]$ is negative, but is about proportional to the $[--c]$ effect for brass in absolute value.

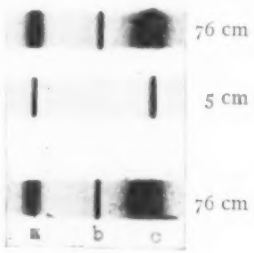
The pole-effect in a brass arc of 6 mm arc gap was measured at currents ranging from 3 amperes to 20 amperes. The results of

these measurements for a member of each triplet series are given in Figure 5. Between 8 and 10 amperes, a line becomes about equal in width at both poles. The pole-effect at the negative pole is but slightly affected by current changes. At 3 amperes the line is narrower at the positive pole than at any other part of the arc, being very sensitive to current changes, broadening unsymmetrically with increased current, until, at 20 amperes, it has shifted 0.09 A. Figure 6 likewise shows the variation of the pole-effect with current for brass. This curve also shows the difference between the pole-effect of a line from a brass arc and from a zinc arc.



12 amps. 5 amps.

Pole-effect at different currents



Pressure-shift at positive pole. Current, 15 amperes.

FIG. 7.—Lines *a* and *b* are not triplet series lines. They are sharp and symmetrical. Line *c* is λ 3072, a line of the series $1p-ms$ of zinc. Pressure-shift, pole-effect, and influence of current are shown by these examples.

POLE-EFFECT AND PRESSURE-SHIFT IN ARC SOURCES OF CALCIUM AND CALCIUM CHLORIDE

Reference has been made to the pressure-shift measurements of the calcium lines by Miller and to the pole-effect measurements of the same lines by Whitney. The difference in the character of the lines from metallic calcium and from calcium chloride observed by Gale and Whitney has also been mentioned. In this work the pole-effects and pressure-shifts of the lines from these two sources were measured under the same conditions and compared.

For the metallic calcium source both terminals were made of calcium rods 0.7 cm in diameter with plane faces. For the calcium

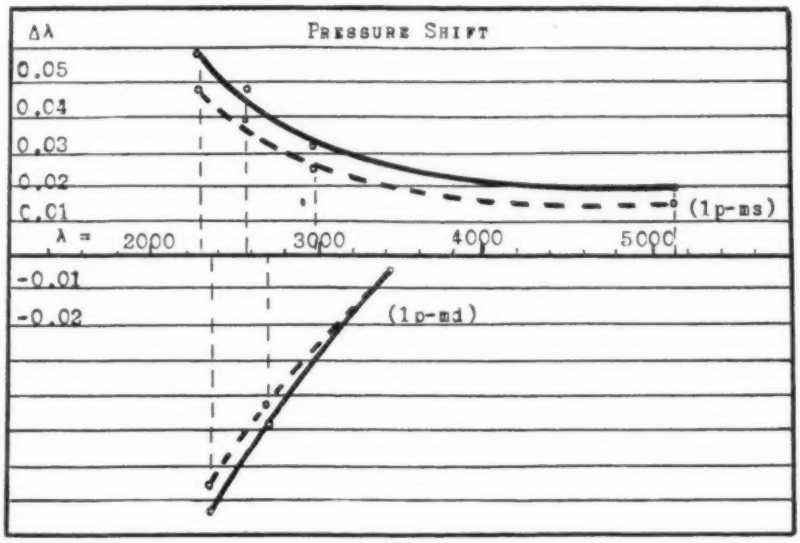


Fig. 8

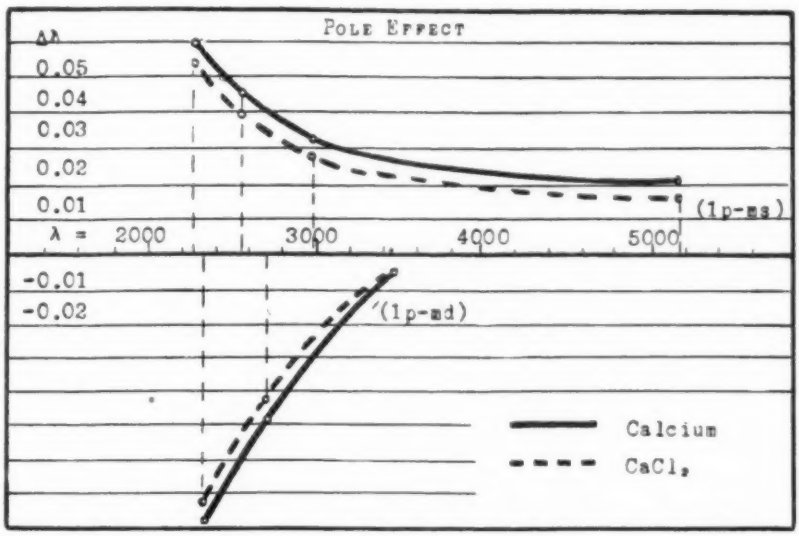


Fig. 9

chloride source the terminals were made of similar carbon rods drilled and filled with a fused mixture of calcium chloride and carbon filings. A 6-ampere current was employed. Exactly the same conditions were used in both sources. The series ($1p-md$) and ($1p-ms$) were the only ones having enough members in the photographic region to give a series relationship.

Triple comparisons were made of lines from the positive pole, the negative pole, and the center of both arc sources. Pressure-shifts were also measured for a range of pressure between 76 cm and 5 cm.

The shifts of the lines of calcium and of calcium chloride were very similar, that of the metallic source being a little greater.

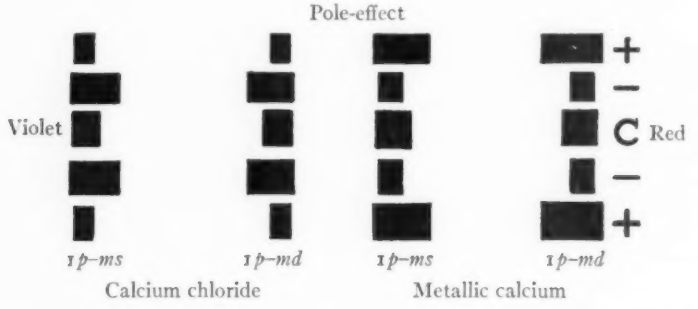


FIG. 10.—Drawn from measurements on a line of the sharp series and of one of the diffuse series from metallic calcium and from calcium chloride arc sources, showing characteristic pole-effects exhibited in each case.

The results here obtained on metallic calcium were consistently larger than obtained by Miller, due to the greater current used in this work.

The measurements of the pole-effect in the lines from the calcium chloride source showed a considerable shift between the center of the arc and the negative pole for both series. As mentioned by Gale and Whitney, the negative pole gives a widened line, the shift at this pole being toward the red for the sharp series, and toward the violet for the diffuse series. However, in the source of metallic calcium the greatest shift occurred at the positive pole and was about the same in magnitude and direction as that at the negative

pole of the calcium chloride source. These results for metallic calcium checked the values obtained by Whitney, except that they were consistently larger.

The results are given in graphical form in Figures 8 and 9. Values from calcium chloride are given in the dotted curves. Figure 8 shows that the pressure-shift of any line is approximately the same in both sources. Figure 9 gives pole-effect data. The full line represents the pole-effect between the positive pole and the center of the arc for metallic calcium. The dotted line gives the effect between the negative pole and the center of the arc for calcium chloride. This difference in the character of the line as given from the two sources is also shown in Figure 10.

The pole-effect displacements were almost identical with the pressure-shifts, the amount of the shift ranging from 0.01 A to 0.06 A. The amount of shift increased with frequency in any series.

RYERSON LABORATORY
June 1923

THE DISTRIBUTION FUNCTIONS FOR STELLAR VELOCITY[1]

By FREDERICK H. SEARES

ABSTRACT

Distribution functions for tangential and radial velocities.—The functions for these velocities are calculated on the *assumption* that the *logarithms of the space-velocities*, corrected for solar motion, are distributed according to a *Gaussian error-curve* about the logarithm of the geometrical mean velocity (Adams, Strömberg, and Joy, *Contribution* No. 210). It is also assumed that the directions of the space-velocities have a random distribution. For stars scattered over the whole sky the error arising from the neglect of systematic motions is negligible (Table VII).

For velocities >10 km/sec. the logarithms of *tangential velocity*, both corrected and uncorrected for solar motion ($\log T$, $\log T_0$), have sensibly a Gaussian distribution (Tables II, III; Figs. 2, 3). Schwarzschild's assumption is therefore justified in the main. For small velocities the frequencies of the velocities themselves are nearly proportional to the velocity.

The corresponding frequency function for *radial velocity*, freed from solar motion, is

$$R(\theta)d\theta \propto d\theta \int_{-\infty}^{h_v x_1} e^{-t^2}dt \qquad x_1 = \log \frac{\theta}{2\bar{\theta}} + \frac{3}{4h_v^2}\ \text{Mod.}$$

where $\bar{\theta}$ is the arithmetical mean velocity without regard to sign, and h_v the modulus of the frequency function for space velocity. The theoretical and observed distributions are in close agreement (Table IV*a*). The deviations from a Gaussian curve (Tables IV, IV*a*, Fig. 4) are not large, but cannot be neglected in accurate work.

Moduli of distribution functions.—Numerical values of the moduli derived from the data in *Contribution* No. 210 are: corrected space-velocity, $h_v = 2.62$; corrected tangential velocity, $h_T = 2.35$; uncorrected tangential velocity, $h = 2.18$. These parameters are nearly if not quite independent of absolute magnitude (Table IV*a*). The value of h is checked by its consistency with the coefficients of the formula for mean parallax (*Contribution* No. 273).

Mean tangential velocity.—The geometrical mean velocity, which enters as a parameter in the distribution function for tangential velocity, depends on absolute magnitude and is difficult to determine because of an influence of selection. For velocities uncorrected for solar motion the relation found (Section 9, Table VI) is

$$\log T_0 = 1.27 + 0.052\ M - 0.0019\ M^2 \quad (T_0 \text{ in km/sec.; Int. scale for } M).$$

The constant term depends on the proper motions and spectroscopic parallaxes of 875 stars, nearly all brighter than $M = 3$, and should be little influenced by selection. The coefficients of the terms in M depend on radial velocities. The use of these minimizes the influence of selection.

Properties of Gaussian error functions.—Various general relations are collected in Section 2.

Relations depending on random direction of space-velocities.—The relations between space, tangential, and radial velocities corresponding to random distribution (Section 3) are independent of the form of the distribution functions of the velocities themselves.

1. INTRODUCTION

In his memoir on the problems of stellar statistics Schwarzschild[2] assumed the logarithms of the linear tangential velocities of

[1] *Contributions from the Mount Wilson Observatory*, No. 272.

[2] *Astronomische Nachrichten*, **190**, 361, 1912.

the stars to be distributed according to a Gaussian error-curve. The assumption was made with respect to velocities uncorrected for solar motion and referred to stars of all luminosities, considered collectively. Meanwhile it has been found that average stellar velocity is a function of absolute magnitude, and it thus becomes desirable to modify Schwarzschild's assumption and write[1]

$$P(\log T_0)d(\log T_0)=\frac{h}{\sqrt{\pi}}e^{-h^2(\log T_0-B)^2}d(\log T_0) \qquad (1)$$

where

$$B=\overline{\log T_0}=\log \underline{T_0}=F(M)\,. \qquad (2)$$

Equation (1) states that the values of $\log T_0$ (T_0 = linear tangential velocity uncorrected for solar motion) for stars of absolute magnitude M have a Gaussian distribution about the mean value $\log \underline{T_0}$ defined by (2). The expression is, therefore, one of a series of component functions whose weighted sum for all values of M is the original distribution function of Schwarzschild. The use of a constant modulus does not overlook the fact that h may vary with M, but there are reasons for believing that any possible change is small.

For the discussion of the luminosity function given in *Contribution* No. 273 it was necessary to establish the validity of (1) and to determine the numerical values of the parameters h and $\log \underline{T_0}$. The investigation is based on the Gaussian distribution of the logarithms of the corrected space-velocities found by Adams, Strömberg, and Joy[2] from a discussion of 1241 stars of types F to M. Although such a law can scarcely hold for very small velocities, they found the representation of the observed data by Gaussian curves to be excellent.

The starting-point is therefore the frequency function

$$F(\log v)d(\log v)=\frac{h_v}{\sqrt{\pi}}e^{-h_v^2(\log v-A)^2}d(\log v) \qquad (3)$$

[1] In *Astronomische Nachrichten*, **198**, 217, 1914, Schwarzschild himself modified his original assumption to the extent of supposing $\log \overline{T_0}$ to be a linear function of M. The distribution function used is not, however, quite that adopted here.

[2] *Mt. Wilson Contr.*, No. 210; *Astrophysical Journal*, **54**, 9, 1921. See particularly Tables VIII and IX and Figure 2.

or its equivalent, giving the distribution of the velocities themselves:

$$F(v)dv=\frac{h_v \text{ Mod.}}{\sqrt{\pi}\, v} e^{-h_v(\log v-A)^2}dv \tag{4}$$

where

$$A=\log \underline{v}=G(M)\,. \tag{5}$$

Relations (3) and (4), it should be emphasized, refer to velocities corrected for the sun's motion.

To find whether the distribution of tangential velocities expressed by (1) is consistent with (3), two steps are necessary: First, the derivation from (3) of the function $P(\log T)$ for *corrected* tangential velocities; second, the transformation of $P(\log T)$ into the function $P(\log T_0)$ for velocities uncorrected for the sun's motion. Comparison of this result with (1) then answers the question of consistency. Incidentally, the distribution functions $R(\theta)$ and $R(\theta_0)$ implied by (3) for corrected and uncorrected radial velocities, respectively, will also be derived.

Logarithms are everywhere to the base 10 unless otherwise indicated.

2. PROPERTIES OF GAUSSIAN ERROR FUNCTIONS

For convenience of reference, a number of general relations connecting the parameters of a Gaussian error function with the various means of the variable are collected here.

For the function

$$y=\frac{h}{\sqrt{\pi}} e^{-h^2x^2},$$

with its maximum at $x=0$,

$$\bar{x}=\frac{1}{h\sqrt{\pi}}, \qquad \overline{x^2}=\frac{1}{2h^2}, \qquad \overline{x^2}=\frac{\pi}{2}\bar{x}^2 \tag{6}$$

where $\bar{x}$ is the arithmetical mean without regard to sign. When y is used as a frequency function, $\sqrt{\overline{x^2}}$ is the dispersion or standard deviation.

The geometrical mean of x is given by

$$\log \underline{x} = \frac{2h \,\text{Mod.}}{\sqrt{\pi}} \int_0^\infty e^{-h^2x^2} \log_e x \, dx$$

which reduces to[1] $\underline{x} = 0.3747/h$; whence $\underline{x}/\bar{x} = 0.664$, or, with sufficient accuracy for most purposes,

$$\underline{x} = \frac{3}{8h} = \tfrac{2}{3}\bar{x} . \tag{7}$$

For the function

$$y = \frac{h}{\sqrt{\pi}} e^{-h^2(x-x_0)^2} ,$$

having its maximum at $x = x_0$

$$\bar{x}^2 = \frac{1}{2h^2} + x_0^2 \tag{8}$$

and the arithmetical mean, without regard to sign, is

$$\bar{x} = \frac{1}{h\sqrt{\pi}} e^{-h^2x_0^2} + x_0 P(hx_0) \tag{9}$$

where

$$P(hx_0) = \frac{2}{\sqrt{\pi}} \int_0^{hx_0} e^{-t^2} dt . \tag{10}$$

For $x_0 = 0$, (8) and (9) reduce to (6).

For the Gaussian function of $\log z$

$$y = \frac{h}{\sqrt{\pi}} e^{-h^2(\log z - \log \underline{z})^2}$$

with its maximum at $\log z = \log \underline{z}$.

$$\log \bar{z} = \log \underline{z} + \frac{1}{4h^2 \,\text{Mod.}} \tag{11}$$

$$\log \bar{z}^2 = 2 \log \bar{z} + \frac{1}{2h^2 \,\text{Mod.}} \tag{12}$$

$$\log \bar{z}^2 = 2 \log \underline{z} + \frac{1}{h^2 \,\text{Mod.}} . \tag{13}$$

[1] Bierens de Haan, *Nouvelles Tables d'Intégrales Définies*, 256, No. 8, 1867.

The second relation is a combination of the first and third, which are derived from the original function by the usual method for calculating means.

3. THE ASSUMPTION OF RANDOM DISTRIBUTION OF SPACE-VELOCITY

It will be assumed that the space-velocities, corrected for solar motion, have a random distribution in direction. The existence of stream motion means that this condition cannot be exactly fulfilled. But since the assumption is used only to derive average results for the stars as a whole, the effect of systematic motions is partially neutralized. Thus used, the assumption leads to no serious error, for the relations ($\bar{\theta}$ = mean radial velocity without regard to sign)

$$\bar{v}=\frac{4}{\pi}\bar{T}=2\bar{\theta} \tag{14}$$

which follow from a strictly random distribution are nearly satisfied. This has been pointed out by Adams, Strömberg, and Joy

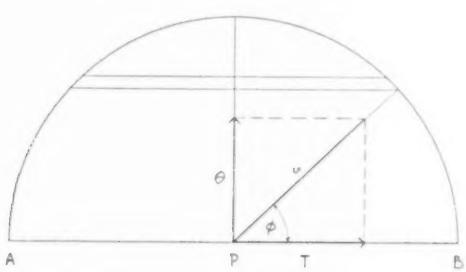


FIG. 1

in *Contribution* No. 210, and is considered in detail in the closing paragraphs of this paper.

Relations connecting the geometrical mean velocities may be derived as follows. If ϕ be the inclination of a space-velocity to the tangential plane of the celestial sphere (Fig. 1),

$$\log \bar{T}=\log \bar{v}+\overline{\log \cos \phi} \tag{15}$$

$$\log \bar{\theta}=\log \bar{v}+\overline{\log \sin \phi}\,. \tag{16}$$

But

$$\overline{\log \cos \phi} = \text{Mod.} \int_0^{\pi/2} \cos \phi \log_e \cos \phi \, d\phi = \log 2 - \text{Mod.}^1 \tag{17}$$

$$\overline{\log \sin \phi} = \text{Mod.} \int_0^{\pi/2} \cos \phi \log_e \sin \phi \, d\phi = -\text{Mod.} \tag{18}$$

whence

$$\log \underline{T} = \log \underline{v} - 0.1333 \tag{19}$$

$$\log \underline{\theta} = \log \underline{v} - 0.4343 \, . \tag{20}$$

Combined with (14), these give

$$\log \overline{T} - \log \underline{T} = \log \bar{v} - \log \underline{v} + 0.0284 \tag{21}$$

$$\log \bar{\theta} - \log \underline{\theta} = \log \bar{v} - \log \underline{v} + 0.1333 \, . \tag{22}$$

Finally, the difference of (15) and (16) gives

$$\underline{T} = 2\underline{\theta} \, . \tag{23}$$

Equation (21), through (11), connects the dispersion in $\log T$ directly with that in $\log v$.

The relations between the mean squares of the corrected and uncorrected velocities follow from

$$\theta_0 = \theta + V \cos \lambda \tag{24}$$

$$T_0^2 = T^2 + V^2 \sin^2 \lambda - 2TV \sin \lambda \cos \alpha \tag{25}$$

where V is the solar velocity, λ the distance from the antapex, and α the angle between the peculiar and parallactic tangential components. The results are

$$\overline{\theta_0^2} = \overline{\theta^2} + \tfrac{1}{3} V^2 \tag{26}$$

$$\overline{T_0^2} = \overline{T^2} + \tfrac{2}{3} V^2 \, . \tag{27}$$

Further

$$\overline{\theta^2} = \tfrac{1}{3}\overline{v^2} \text{ and } \overline{T^2} = \tfrac{2}{3}\overline{v^2} \, , \tag{28}$$

hence

$$\overline{T_0^2} = 2\overline{\theta_0^2} \, . \tag{29}$$

[1] Bierens de Haan, *op. cit.*, **306**, No. 3, 1867.

All of the relations of this section follow from the single assumption of random distribution of the directions of the corrected space-velocities, and are independent of the form of the frequency functions of the various velocities.

4. FORMULA FOR $P(\log T)$ AND $R(\theta)$

Let the space-velocities be represented in direction and magnitude by lines radiating from a point P (Fig. 1). The radial and tangential velocities corresponding to any v are

$$T = v \cos \phi \qquad \theta = v \sin \phi \,. \tag{30}$$

For velocities above the plane AB the probability of a v having an inclination between ϕ and $\phi + d\phi$ is $\cos\phi \, d\phi$ (area of zone of width $r d\phi$ divided by area of hemisphere of radius r). Further, the probability of a speed between v and $v+dv$ is $F(v)\,dv$. Since the directions of v are assumed to be random, speed and direction are independent. Hence the probability that the T or θ defined by (30) arises from a pair of values of v and ϕ within the limits specified is $F(v)dv \cdot \cos\phi \, d\phi$. Since other combinations of ϕ and v lead to the same values of T and θ, the total probabilities, for the intervals T to $T+dT$ and θ to $\theta+d\theta$, respectively, are

$$P(T)dT = \int_0^{\pi/2} F(v)dv \cos\phi \, d\phi \tag{31}$$

$$R(\theta)d\theta = \int_0^{\pi/2} F(v)dv \cos\phi \, d\phi \tag{32}$$

where the integration is with respect to ϕ. In the first integral $F(v)dv$ must be expressed as a function of T and ϕ; in the second, as a function of θ and ϕ.

The v's below the plane AB also lead to the integrals (31) and (32); but the radial velocities are negative. Hence $R(\theta)$ is a symmetrical function, with $R(+\theta) = R(-\theta)$.

The combination of (4) and (30) in accordance with (31) and (32) gives[1]

[1] The subscript for the modulus h is often omitted when there is no uncertainty as to the function to which it refers.

$$P(T)dT=\frac{dT\,h\ \text{Mod.}}{T\sqrt{\pi}}\int_0^{\pi/2}e^{-h^2(L-\log\cos\phi)^2}\cos\phi\,d\phi \tag{33}$$

$$R(\theta)d\theta=\frac{d\theta\,h\ \text{Mod.}}{\theta\sqrt{\pi}}\int_0^{\pi/2}e^{-h^2(N-\log\sin\phi)^2}\cos\phi\,d\phi \tag{34}$$

where

$$L=\log T-A \qquad N=\log\theta-A. \tag{35}$$

The first integral is treated by quadratures in Section 6. The second can be simplified by the substitution $x=\log\sin\phi$, whence

$$\text{Mod.}\cos\phi\,d\phi=e^{\frac{x}{\text{Mod.}}}dx$$

and

$$R(\theta)\,d\theta=\frac{d\theta}{\theta}\frac{h}{\sqrt{\pi}}\int_{-\infty}^{0}e^{-h^2(N-x)^2+\frac{x}{\text{Mod.}}}dx\,.$$

This may be written

$$R(\theta)d\theta=\frac{d\theta}{\theta}\frac{h}{\sqrt{\pi}}\,e^{p}\int_{-\infty}^{0}e^{-h^2(x-x_1)^2}\,dx \tag{36}$$

in which

$$p=\frac{1}{\text{Mod.}}\left(N+\frac{1}{4h^2\,\text{Mod.}}\right) \qquad x_1=\left(N+\frac{1}{2h^2\ \text{Mod.}}\right). \tag{37}$$

Since the distribution of $\log v$ is Gaussian, (11) gives

$$\frac{1}{4h^2\ \text{Mod.}}=\log\bar{v}-\log v\,,$$

and since $A=\log v$, (37) reduces, with the aid of (35), to

$$p=\frac{1}{\text{Mod.}}\log\frac{\theta\bar{v}}{(v)^2} \qquad x_1=\log\theta\,\frac{\bar{v}^2}{(v)^3}=\log\frac{\theta}{2\bar{\theta}}+\frac{3}{4h_v^2\ \text{Mod.}}\,. \tag{38}$$

Hence $e^p=\theta\bar{v}/v^2$. Finally, writing $t=h_v(x-x_1)$, (36) becomes

$$R(\theta)d\theta=d\theta\frac{\bar{v}}{(v)^2}\frac{1}{\sqrt{\pi}}\int_{-\infty-h_vx_1}^{-h_vx_1}e^{-t^2}\,dt\,. \tag{39}$$

Equation (34) thus depends upon the probability integral whose values are known. The lower limit of (39), indeterminate for $\theta=0$, is $-\infty$ for all finite values of θ, however small.

Both (33) and (39) are frequency functions for the velocities themselves. The latter is conveniently used as it stands, but it is desirable to replace (33) by the corresponding function for the logarithms, namely,

$$P(\log T)d(\log T) = d(\log T)\,\frac{h_v}{\sqrt{\pi}}\int_0^{\pi/2} e^{-h_v^2(\log T - \log\cos\phi - A)^2}\cos\phi\,d\phi\,. \quad (40)$$

5. OBSERVATIONAL DATA

To determine the parameters h_v and $A = \log \bar{v}$ occurring in (39) and (40) observational data are required. Those used (Table I) are a combination, according to absolute magnitude, of the results for about 1300 stars of spectral types F to M given in Table I of *Contribution* No. 210. Stars having space-velocities greater than 180 km/sec. were excluded, since they belong to a group not representative of the stars as a whole. Velocities with the subscript o are uncorrected for solar motion; the others have been corrected.

The data in the first, third, and sixth columns of the table are a numerical expression of relations (5) and (2), respectively. The latter is discussed in detail in Section 9. The third column gives directly the value of A for any M.

TABLE I

MEAN STELLAR VELOCITY AND ABSOLUTE MAGNITUDE BASED ON RESULTS OF ADAMS, STRÖMBERG, AND JOY, TABLE I, *Contribution* No. 210

M	$\log \bar{v}$	$\log \underline{v}$	$\log \bar{v}/\underline{v}$	$\log \bar{T}_o$	$\log \underline{T}_o$	$\log \bar{T}_o/\underline{T}_o$	θ	$\log T_o$ from θ	Wt.
− 2.1..	1.243	1.192	0.051	1.173	1.126	0.047	9.4	1.172	3.2
− 0.8..	1.487	1.354	.133	1.467	1.264	.203	11.5	1.224	6
+ 0.2..	1.530	1.434	.096	1.442	1.270	.172	16.8	1.342	15
0.9..	1.556	1.470	.086	1.459	1.328	.131	21.0	1.422	19
1.9..	1.518	1.438	.080	1.435	1.289	.146	18.5	1.376	15
3.0..	1.585	1.524	.061	1.545	1.453	.092	18.8	1.382	9
4.1..	1.754	1.667	.087	1.727	1.620	.107	24.4	1.479	13
5.8..	1.807	1.727	.080	1.809	1.741	.068	25.9	1.502	17
8.4..	1.724	1.671	.053	1.680	1.633	.047	30.1	1.562	2
+10.2..	1.791	1.740	0.051	1.715	1.654	0.061	37.9	1.656	2
Means			0.084			0.121			

Since the distribution of $\log v$ for each value of M is known from the results of *Contribution* No. 210 to be Gaussian, the differences

$\log \bar{v} - \log \underline{v}$ in the fourth column of Table I can be used with (11) to determine the modulus for successive values of M. The question whether this parameter really varies with M recalls that raised in connection with the modulus for $P(\log T_0)$. The dispersion in v certainly increases with M; but this does not necessarily imply a change in the dispersion in $\log v$, since by (6)

$$\frac{1}{2h_v^2} = \overline{(\log v - \log \underline{v})^2} . \tag{41}$$

Thus, if the velocities of all frequencies increase in the same ratio with increasing M, the value of h_v will be independent of M.

The data in Table I suggest a decrease in the dispersion in $\log v$ as M increases; but this cannot be accepted without reservation, because the stars used are affected by a selection favoring large space-velocities. The observed dispersion for large values of M is, therefore, too small. The influence of selection cannot be estimated, but it seems likely that any real change in h_v is slight, and that the value corresponding to the weighted mean $\log \bar{v}/\underline{v} = 0.084$ can be used for all values of M without serious error. We thus find

$$h_v = 2.62 . \tag{42}$$

6. DISTRIBUTION OF TANGENTIAL VELOCITIES CORRECTED FOR SOLAR MOTION

Since only the form of the distribution function $P(\log T)$ is required, the constant factor in (40) can be disregarded. For the computation write, therefore,

$$\log \{P(\log T)\} = \Sigma\{-\text{Mod.}\, h^2(\log T - \log \cos \phi - \log \underline{v})^2 + \log \cos \phi\} \tag{43}$$

where the summation is for values of ϕ from 0° to 90°.

To test the validity of the modified assumption of Schwarzschild expressed by (1) it will be sufficient to carry through the calculation for a single value of M. If it holds for this case it will certainly hold for other values of M. On the basis of Table I, $\log \underline{v} = 1.50$, ($M = 3$) is adopted as representative. Then, with (42), (43) becomes

$$\log \{P(\log T)\} = \Sigma\{-[1.73(\log T - 1.50) - 1.73 \log \cos \phi]^2 + \log \cos \phi\}. \tag{44}$$

This expression has been summed for $\phi = 5°, 15° \ldots\ldots 85°$, for values of $\log T$ differing by 0.2. The results are in the second column of Table II. The resulting frequency curve for $\log T$ (Fig. 2) is slightly asymmetrical, but capable of good representation by a Gaussian curve (points, Fig. 2) having the constants

$$\log \underline{T} = 1.367 \qquad h_T = 2.24 . \tag{45}$$

TABLE II

DISTRIBUTION OF TANGENTIAL VELOCITIES CORRECTED FOR SOLAR MOTION

$\log T$	$P(\log T)$	$P(\log T)_G$	Residual	$\log T$	$P(\log T)$	$P(\log T)_G$	Residual
+0.0.....	0.001	.....	+0.001	+1.6.....	0.210	0.210	0.000
0.2.....	.004	.....	.004	1.8.....	.103	.112	− .009
0.4.....	.007	0.001	.006	2.0.....	.030	.038	.008
0.6.....	.016	.009	.007	2.2.....	.005	.009	− .004
0.8.....	.046	.038	+ .008	+2.4.....	0.001	0.001	0.000
1.0.....	.112	.112	.000				
1.2.....	.205	.210	− .005	Sums...	1.000	1.000	+0.026
+1.4.....	0.260	0.260	0.000				−0.026

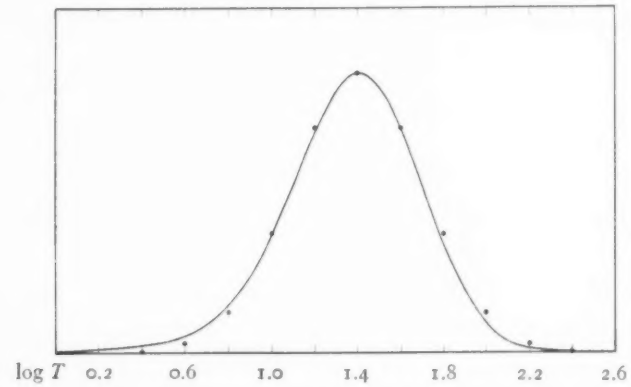


FIG. 2.—The ordinates of the curve represent the frequencies of the logarithms of the tangential velocities, corrected for solar motion, which correspond to an error-curve distribution of the logarithms of the space-velocities about the mean $\log v = 1.50$. The points indicate a Gaussian error-curve.

The first is the mean of the values of $\log T$ weighted according to the frequencies in the second column of Table II. The modulus can be found by (8), but more satisfactorily by the first of (6), x representing the differences $\log T - \log \underline{T}$. Since random distribution of

velocities has been assumed, the entire calculation is checked by (19), which reproduces the foregoing value of log $\underline{T}$ exactly; further, since $P(\log T)$ is very nearly Gaussian in form, the modulus can also be found by (21) and (11). The first of these gives

$$\log \bar{T} - \log \underline{T} = 0.112 , \tag{46}$$

whence $h_T = 2.27$, in satisfactory agreement with the value from the curve.

Owing to the appreciable skewness of $P(\log T)$, a slight modification of the Gaussian constants improves the representation. Those used to derive the ordinates in the third column of Table II are

$$\log \underline{T} = 1.40 \qquad h_T = 2.35 . \tag{47}$$

The sum of the corresponding deviations is 5 per cent of the area under the curve, but the individual values are so distributed that much observational material would be required to establish their reality. For practical purposes, therefore, it can usually be assumed that the logarithms of the tangential velocities, freed from solar motion, have a Gaussian distribution.

7. DISTRIBUTION OF TANGENTIAL VELOCITIES UNCORRECTED FOR SOLAR MOTION

The second step of the transition from $F(\log v)$ to $P(\log T_0)$ is based on the general relation (25). A complete solution would require a double integration: with respect to α, the inclination of the peculiar velocity T to the great circle connecting a star with the antapex; and with respect to λ, the distance from the antapex. The first integration must be performed, but the result of the second can be inferred from the character of the function found by integrating with respect to α for a special value of λ.

For $\lambda = 0$ or 180°, $T_0 = T$, and $P(\log T_0)$ reduces to $P(\log T)$. The maximum distortion of this function resulting from the inclusion of the sun's motion occurs for $\lambda = 90°$. The transformation is therefore effected for this special case. Equation (25) accordingly becomes

$$T_0^2 = T^2 + V^2 - 2TV \cos \alpha . \tag{48}$$

To simplify the computation an even 20 km/sec. was adopted for the solar velocity, although a slightly different value had been used in reducing the observational data for *Contribution* No. 210.

Random distribution of the directions of v implies random distribution of the values of α. The probability of an angle between α and $\alpha+d\alpha$ is proportional to $d\alpha$, and is independent of the probability of T. Combining these probabilities, we find for the frequency function of $\log T_0$

$$P(\log T_0)d(\log T_0)=d(\log T_0)\int_0^\pi P(\log T)\frac{d(\log T)}{d(\log T_0)}d\alpha\,. \qquad (49)$$

The essential part of this expression may be written

$$P(\log T_0)=T_0\int_0^\pi P(T)\frac{dT}{dT_0}d\alpha$$

where

$$P(T)=P(\log T)/T\,.$$

From (48)

$$T=\pm\sqrt{T_0^2-V^2\sin^2\alpha}+V\cos\alpha \qquad (50)$$

whence

$$\frac{dT}{dT_0}=\pm\frac{1}{\sqrt{1-k^2\sin^2\alpha}}, \qquad k=\frac{V}{T_0}, \qquad (51)$$

the sign of the radical changing for $T_0=V\sin\alpha$. We thus have

$$P(\log T_0)=T_0\int_0^\pi\frac{P(T)}{\sqrt{1-k^2\sin^2\alpha}}d\alpha\,. \qquad (52)$$

$P(T)$ is by (50) a function of T_0 and α, known only through its numerical values based on the data in Table II. Even were $P(\log T)$ replaced by the Gaussian error-function which represents it closely, it would be impossible to handle (52) analytically. On the other hand, a numerical integration is complicated by the

fact that the element of the integral for $k^2\sin^2\alpha = 1$ is infinite. The difficulty is avoided by replacing (52) by a sum of integrals:

$$P(\log T_0) = T_0 \sum_0^{\pi} P(T)_m \int_{\alpha_1}^{\alpha_2} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}} \tag{53}$$

or

$$P(\log T_0) = T_0 \sum_0^{\pi} \frac{P(T)_m}{\cos\alpha_m} \int_{\alpha_1}^{\alpha_2} \frac{\cos\alpha\, d\alpha}{\sqrt{1-k^2\sin^2\alpha}} \tag{53a}$$

where the interval $\alpha_2-\alpha_1$ is so small that variations in $P(T)$ and $\cos\alpha$ can be neglected. It is thus possible to replace these factors by mean values written outside the integral as constants.

It is convenient to distinguish two cases: $k<1$ and $k>1$. For the first

$$\int_{\alpha_1}^{\alpha_2} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}} = F(k, \alpha_2) - F(k, \alpha_1)\,, \tag{54}$$

the values of F being given by tables of elliptic integrals. For the second, the alternative form (53*a*) gives

$$\int_{\alpha_1}^{\alpha_2} \frac{\cos\alpha\, d\alpha}{\sqrt{1-k^2\sin^2\alpha}} = \frac{1}{k}\left[\sin^{-1}(k\sin\alpha)\right]_{\alpha_1}^{\alpha_2}. \tag{55}$$

For $k=1$ both integrals are infinite for $\alpha=90°$. The expression for $P(\log T_0)$ then becomes indeterminate because $T_0=V$, $T=0$, and, theoretically at least, $P(T)=0$. The value of $P(\log T_0)$ is easily found, however, by interpolation from adjacent values for which there is no uncertainty in the calculation.

The procedure for the evaluation of (53) and (53*a*) is as follows: For equidistant values of $\log T_0$ (except for two or three critical regions an interval of 0.2 was used) calculate T from (50) for values of α between 0 and π. Generally speaking, 10° intervals are

sufficient. With T as argument take $P(T)$ from a table based on the data in the first two columns of Table II. Then for $k<1$

$$P(\log T_0)=T_0\sum P(T)_m\{F(k, a_2)-F(k, a_1)\} \tag{56}$$

and for $k>1$

$$P(\log T_0)=\frac{T_0}{k}\sum\frac{P(T)_m}{\cos a_m}\{\sin^{-1}(k \sin a_2)-\sin^{-1}(k \sin a_1)\}. \tag{57}$$

The actual calculation is therefore simple. The results are in the second column of Table III, and are illustrated in Figure 3. In addition, both table and diagram show the representation of the data by a Gaussian curve (points, Fig. 3) having the constants

$$\log T_0=1.51 \qquad h=3.00. \tag{58}$$

TABLE III

DISTRIBUTION OF TANGENTIAL VELOCITIES UNCORRECTED FOR SOLAR MOTION ($\lambda=90°$)

log T_0	P (log T_0)	P (log T_0)$_G$	Residual	log T_0	P (log T_0)	P (log T_0)$_G$	Residual
0.0.....	0.001	0.000	+0.001	1.5.....	0.327	0.319	+0.008
0.1.....	.001	.000	.001	1.6.....	.300	.296	+ .004
0.2.....	.002	.000	.002	1.7.....	.230	.232	− .002
0.3.....	.003	.000	.003	1.8.....	.146	.152	.006
0.4.....	.004	.000	.004	1.9.....	.080	.083	− .003
0.5.....	.006	.000	.006	2.0.....	.038	.038	.000
0.6.....	.010	.000	.010	2.1.....	.016	.014	+ .002
0.7.....	.015	.001	.014	2.2.....	.008	.006	.002
0.8.....	.024	.004	.020	2.3.....	.004	.001	.003
0.9.....	.034	.012	.022	2.4.....	.001	.000	.001
1.0.....	.049	.031	.018	2.5.....	0.001	0.000	+0.001
1.1.....	.078	.072	.006				
1.2.....	.140	.133	+ .007	Sums...	2.000	1.897	+0.135
1.3.....	.205	.216	− .011				−0.032
1.4.....	0.277	0.287	−0.010				

Except for the small velocities, which are of low frequency, the agreement is excellent; for general statistical discussions the approximation of the Gaussian curve is ample. Thus the total deviations are 8 per cent of the area, and 5 of the 8 per cent are associated with velocities less than 10 km/sec., which include but 7 per cent of all the velocities. The large excess of positive residuals is intentional, because it corresponds to a better representation of the velocities of high frequency.

The value of log $\underline{T}_0$ can be checked by (27), which for the special case $\lambda = 90°$ and $V = 20$ km/sec. becomes

$$\overline{T_0^2} = \overline{T^2} + 400 . \tag{59}$$

The mean squares are connected with the constants of the Gaussian functions representing $P(\log T)$ and $P(\log T_0)$ by (13). The values in (47) and (58) are not quite consistent with (59), but this condition becomes satisfied if log $\underline{T}$ is modified by -0.02 and log $\underline{T}_0$ by $+0.01$. Figures 2 and 3 show that displacements of the Gaussian curves by quantities of this order are necessary to represent

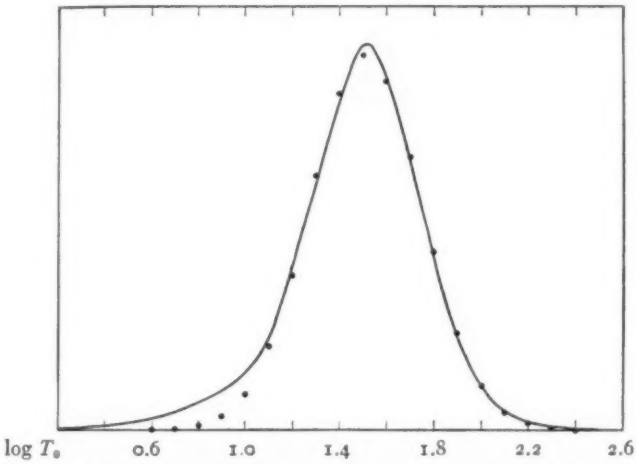


FIG. 3.—The curve represents the distribution of the logarithms of the tangential velocities, uncorrected for solar motion, at points 90° from the sun's apices. The basis of the calculation is the same as for Fig. 2. The points indicate a Gaussian error-curve.

accurately the largest velocities, which are of dominating influence in calculating the mean squares. The small velocities contribute so little to the sum of the squares that, practically speaking, their frequencies escape the test altogether, but these are sufficiently controlled by the run in the differences.

Since the form of $P(\log T_0)$ for $\lambda = 90°$, which presents the maximum deviation from $P(\log T)$, is very nearly Gaussian, it follows that Gaussian curves can be used for all other values of λ. The complete distribution function is the weighted mean of all such curves, the weighting factor in each case being sin λ. Since the

abscissae of the maxima of all the curves fall within the limits 1.40 and 1.51, the progressive change in the values of the weighting factor can have only the slightest influence on the symmetry of the resultant curve. As an illustration, consider the similar case presented by the calculation of $P(\log T)$ by (40), which represents the sum of a series of Gaussian curves with the weighting factor $\cos\phi$. The curves which contribute appreciably to the resultant have their maxima scattered over an interval many times that occurring in the case of $\log T_0$, and yet the resulting asymmetry is only that shown by Figure 2.

In the case of $P(\log T_0)$ the asymmetry of the weighted mean function cannot be appreciably greater than that for the extreme case $\lambda = 90°$, illustrated by Figure 3. Further, the moduli of the component curves which represent $P(\log T_0)$ for different values of λ lie within the limits $h = 2.35$ (Fig. 2) and 3.00 (Fig. 3). The ranges in both parameters are therefore small. This circumstance, together with the very slight asymmetry of the resultant curve, shows, without actual integration with respect to λ, that $P(\log T_0)$ for the whole sky must be very approximately a Gaussian curve. The deviations will be of the same order as those shown in Figures 2 and 3, and will be associated chiefly with the velocities of low frequency.

For most purposes, therefore, the distribution of uncorrected tangential velocities expressed by (1) can be regarded as consistent with a Gaussian distribution of the logarithms of the space-velocities, and, hence, as valid when the stars are considered collectively.[1] But under circumstances such that the smaller velocities (less than about 10 km/sec.) are of dominating influence the Gaussian frequency function does not apply, and its use will lead to erroneous results. An examination of the data in Table III ($\lambda = 90°$) shows that for values of $T_0 < 10$ km/sec. the frequencies of the velocities themselves are well represented by

$$F(T_0)dT_0 \propto T_0^n dT_0 .$$

The same formula is applicable to the small values of T in Table II

[1] Luyten, *Proceedings National Academy of Sciences*, **9**, 191, 1923, finds that the observational data for about a hundred stars within 10 parsecs of the sun are in agreement with this result.

(which are the values of T_0 at the apices), whence it may be inferred that an expression of this form can be used for all values of λ. For these data η is approximately 1; but it must be remembered that the assumed distribution for space-velocity also fails for small velocities.

8. DISTRIBUTION OF RADIAL VELOCITIES

The calculation of the distribution of the radial velocities by (39) requires no comment other than the remark that, as before, $h_\tau = 2.62$, $\log \varpi = 1.50$, and $\log \bar{v} - \log \varpi = 0.084$. The results are in the second column of Table IV, and are illustrated in Figure 4. Equation (14) gives $\bar{\theta} = 19.2$ km/sec., which agrees with the weighted mean of the values in the table and checks the calculation.

TABLE IV

DISTRIBUTION OF RADIAL VELOCITIES

Velocity	$R(\theta)$	$R(\theta)_G$	Residual	$R(\theta_e)$		$R_1(\theta_e)$	$R_1(\theta_e)_G$	Residual
km/sec.								
2.5.....	0.192	0.190	+0.002	0.087	0.078	0.165	0.167	−0.002
7.5.....	.184	.180	+ .004	.091	.066	.157	.160	.003
12.5.....	.155	.161	− .006	.091	.054	.145	.146	− .001
17.5.....	.121	.134	.013	.087	.041	.128	.128	.000
22.5.....	091	.107	.016	.078	.030	.108	.107	+ .001
27.5.....	.066	.080	.014	.065	.022	.087	.086	.001
32.5.....	.048	.057	.009	.054	.013	.067	.066	.001
37.5.....	.036	.037	− .001	.041	.008	.049	.048	.001
42.5.....	.026	.024	+ .002	.030	.005	.035	.034	.001
47.5.....	.019	.014	.005	.021	.003	.024	.023	+ .001
52.5.....	.014	.008	.006	.013	.002	.015	.015	.000
57.5.....	.011	.004	.007	.008	.001	.009	.009	.000
62.5.....	.008	.002	.006	.005	.....	.005	.005	.000
67.5.....	.006	.001	.005	.003	.....	.003	.003	.000
72.5.....	.005	.001	.004	.002	.....	.002	.002	.000
77.5.....	.004	.....	.004	.001	.....	.001	.001	.000
82.5.....	.003	.....	.003					
87.5.....	.002	.....	.002					
92.5.....	.002	.....	.002					
97.5.....	.001	.....	.001					
>97.5.....	.006	.....	+ .006					
Sums...	1.000	1.000	+0.059 −0.059	0.677	0.323	1.000	1.000	+0.006 −0.006

No very exact representation of $R(\theta)$ by a Gaussian function can be expected, and the deviations from an error-curve (modulus $k = 0.0341$) shown in the fourth column of Table IV are smaller than might have been anticipated. The sum of the residuals

without regard to sign corresponds to 12 per cent of the area, with a maximum of 1.6 per cent for a 5-km interval in θ. A Gaussian representation will, therefore, sometimes be useful for approximate investigations. Although the frequencies of θ are fairly well represented, the arithmetical mean velocity of 16.5 km/sec. corresponding to the Gaussian curve does not agree well with the foregoing value of 19.2. The difference illustrates the characteristic feature of

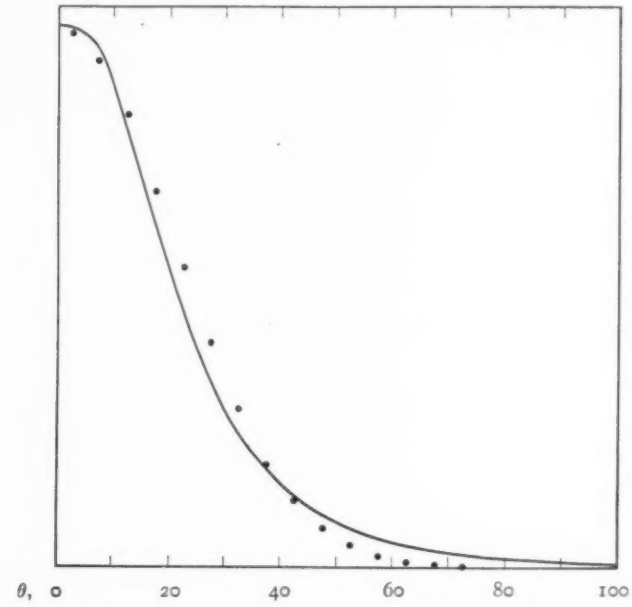


FIG. 4.—The curve represents the distribution of radial velocities, corrected for solar motion, corresponding to a Gaussian distribution of the logarithms of the space-velocities (log $v = 1.50$). The points indicate an error-curve.

the actual distribution, which is a considerable excess of large velocities over the number required by a Gaussian function.

That the distribution of observed radial velocities is really in agreement with (39) is easily shown by comparison with the data for several hundred stars (corrected for solar motion) collected by van Rhijn in Table 3 of *Groningen Publication* No. 34. The weighted means of his two series for $m = 5$ are given in the second column of Table IV*a* below. The corresponding $\bar{\theta}$ is 12.65 km/sec.,

and since the stars are scattered over the whole sky, we can safely assume that the arithmetical mean space-velocity is twice this amount. Using therefore $\bar{v}=25.3$ km/sec. and $h_v=2.62$, we find the deviations from (39) shown in the third column of Table IV*a*. These are obviously of the order of the irregularities in the observed data and show no evidence of systematic divergence. Their superiority to the residuals given by Gaussian curves may be seen by reference to the fourth and fifth columns of the table. The first of these refers to a curve having a modulus calculated in the usual way from $\bar{\theta}$. The second series of residuals corresponds to an adjustment which somewhat improves the representation for the velocities of high frequency.

TABLE IV*a*

COMPARISON OF EQUATION (39) WITH OBSERVED DISTRIBUTION OF RADIAL VELOCITIES

MEDIAN θ	OBS. $R(\theta)$	O.−C.			OBS. $R(\theta)$		O.−C.	
		Eq. (39)	Gaussian Curves		$M=-1.5$	$M=+3.5$	$M=-1.5$	$M=+3.5$
2.5	0.296	+0.006	+0.046	+0.007	0.33	0.19	−0.01	−0.01
7.5	0.255	+0.009	+0.030	+0.003	0.26	0.22	−0.01	+0.03
12.5 . . .	0.147	−0.022	−0.038	−0.046	0.16	0.13	0.00	−0.03
17.5 . . .	0.105	−0.002	0.032	−0.023	0.12	0.11	+0.02	0.01
25.0 . . .	0.120	+0.011	−0.027	+0.008	0.08	0.14	0.00	−0.01
37.5 . . .	0.054	0.000	+0.002	0.029	0.04	0.11	0.00	+0.01
52.5 . . .	0.011	−0.004	0.007	0.010	0.00[3]	0.04	−0.01	0.00
>60 . . .	0.012	+0.004	+0.012	+0.012	0.00[3]	0.06	0.00	+0.01

Two other comparisons, made in precisely the same way, are also given. These relate to two groups of stars from van Rhijn's data whose mean absolute magnitudes differ by five magnitudes. The residuals in the last two columns of Table IV*a* show excellent agreement with (39). The significant point here is that the same value of the modulus for the logarithms of the space-velocities has been used throughout. Since there is no trace of systematic divergence, it is clear that any dependence of h_v upon M must be small.

These comparisons check in a very satisfactory way the distribution function for corrected radial velocities expressed by (39), and give added confidence in the adopted frequency law upon which the whole investigation is based, namely, the Gaussian distribution of the logarithms of the space-velocities.

The frequency function for radial velocities uncorrected for solar motion is of minor importance, although it is sometimes useful to know its general characteristics. These may be found with sufficient accuracy from

$$R(\theta_0)d\theta_0 = d\theta_0 \int_0^\pi R(\theta) \frac{d\theta}{d\theta_0} \sin \lambda d\lambda \tag{60}$$

by supposing $R(\theta)$ to be an error function. Introducing θ_0 by (24), we then have

$$R(\theta_0)d\theta_0 = d\theta_0 \frac{k}{\sqrt{\pi}} \int_0^\pi e^{-k^2(\theta_0 - V\cos\lambda)^2} \sin \lambda d\lambda$$

which by the substitution $t = k(\theta_0 - V\cos\lambda)$ reduces to

$$R(\theta_0)d\theta_0 = d\theta_0 \frac{1}{\sqrt{\pi}\, V} \int_{k(\theta_0 - V)}^{k\theta_0} e^{-t^2} dt\,. \tag{61}$$

Geometrically, $R(\theta_0)$ is proportional to the mean ordinate of an error-curve between two abscissae differing by the constant kV. Since V is small as compared with the range in θ_0, $R(\theta_0)$ must be very nearly a Gaussian curve. The maximum of (61) corresponds to $\theta_0 = V/2$. Hence write

$$R(\theta_0)d\theta_0 = \frac{k_0}{\sqrt{\pi}} e^{-k_0^2(\theta_0 - V/2)^2} d\theta_0\,. \tag{62}$$

By (8), (26), and the second of (6),

$$\overline{\theta_0^2} = \frac{1}{2k_0^2} + \tfrac{1}{4}V^2 = \frac{1}{2k^2} + \tfrac{1}{3}V^2 \tag{63}$$

whence

$$\frac{1}{2k_0^2} = \frac{1}{2k^2} + 1\tfrac{1}{2}V^2 \tag{64}$$

which determines the modulus k_0.

The function $R(\theta_0)$ presupposes that the signs of the velocities have been retained. Since $V/2$ is only 10 km/sec., the frequencies without regard to sign, found by combining the values of $R(+\theta_0)$ and $R(-\theta_0)$, are also nearly Gaussian. The corresponding function

$R_1(\theta_0)$ has its maximum at $\theta_0=0$, and terminates (or begins) with this value. We write accordingly

$$R_1(\theta_0)d\theta_0=\frac{k_1}{\sqrt{\pi}}e^{-k_1^2\theta_0^2}d\theta_0\,. \tag{65}$$

Since the mean square velocity is the same as for (62), the modulus k_1 is connected with k_0 and k by equations (63) and

$$\frac{1}{2k_1^2}=\overline{\theta_0^2}\,. \tag{66}$$

For numerical illustration we start with the Gaussian curve in the third column of Table IV which best represents the values of $R(\theta)$. The corresponding values of $R(\theta_0)$ calculated by the exact formula (61) are in the fifth and sixth columns of Table IV, the latter giving the frequencies for negative velocities. The sums of quantities on the same line, which represent the frequencies $R_1(\theta_0)$ without regard to sign, are in the seventh column.

The modulus k of $R(\theta)$ is 0.0341; hence by (64) $k_0=0.0329$. The resulting values of $R_0(\theta)$ from (62) reproduce those from (61) in all cases but two, where the differences are a single unit of the third place of decimals. Finally, (66) and (63) give $k_1=0.0298$. With this modulus (65) represents the values of $R_1(\theta_0)$ in the seventh column of Table IV with residuals as shown in the last column, the total deviation being but 1 per cent of the area under the curve. The closeness of the representation is further illustrated by the three values of $\overline{\theta_0}$: 19.1, 19.0, 18.9. The first is the direct mean, with the data in the seventh column of Table IV as weights; the second is calculated by (9) with the modulus k_0; the third by the first of (6), using k_1. The relation between the three curves is shown in Figure 5.

The distribution of the corrected and uncorrected velocities is therefore essentially the same except for the difference in dispersion implied by the change from k to k_0 defined by

$$\frac{1}{2k_0^2}=\frac{1}{2k^2}+1\tfrac{1}{2}\,V^2\,.$$

This result assumes that $R(\theta)$ can be represented by a Gaussian function; but in a general way the conclusion holds also for the frequency functions which accurately represent the actual velocities.

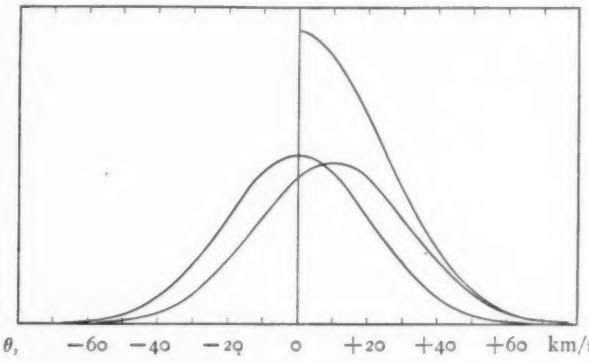


FIG. 5.—Relation between frequencies of radial velocities corrected and uncorrected for solar motion. For simplicity it is assumed that the corrected velocities are represented by an error-curve, symmetrical about $\theta=0$. For a solar velocity of 20 km/sec. the corresponding distribution of the uncorrected velocities is very nearly Gaussian, with a slightly increased dispersion, about the mean $\theta=+10$ km/sec. The distribution of the uncorrected velocities without regard to sign (the curve beginning at $\theta=0$) is also sensibly Gaussian.

9. THE PARAMETERS OF THE DISTRIBUTION FUNCTION FOR UNCORRECTED TANGENTIAL VELOCITY

The results of Section 7 show that the Gaussian distribution (1) can usually be adopted for the values of $\log T_0$. The parameters h and $\log T_0$ are still to be determined. The numerical values of these quantities are important because $P(\log T_0)$ is one of the three fundamental distribution laws upon which nearly all the problems of stellar statistics depend.

Values of h are known for two special cases, 2.35 from (47) for $\lambda=0$, and 3.00 from (58) for $\lambda=90°$. The value for the whole sky must be less than either of these, since the integration over λ necessarily increases the dispersion. As explained in Section 7, this integration has not been performed, so that a calculated value based on the assumed dispersion in space-velocity is not available. The data of Table I, however, give the required result directly, for since the weighted mean of the quantities in the seventh column is

$$\log \bar{T}_0 - \log \underline{T}_0 = 0.121 \qquad (67)$$

h, by (11), is found to be 2.18. In spite of the possible influence of selection, this result is probably near the truth, because in *Contribution* No. 273 it is shown to be consistent with the constants of the formula for the mean parallax.

For the present we must assume that the modulus is independent of M, which may not be wholly justifiable. The individual differences, of which (67) is the mean, show a pronounced tendency to decrease with increasing $\overline{M}$. The circumstances are similar to those for the space-velocities referred to in Section 5, but with the added argument that a change in $\log \overline{T}_0/\underline{T}_0$ as large as suggested by Table I is inconsistent with the known relation between mean parallax, proper motion, and apparent magnitude.

Owing to troublesome systematic errors, the derivation of $\log \underline{T}_0$ as a function of M is not a simple matter. Provisional results have been given by Adams, Strömberg, and Joy in *Contribution* No. 210. Some further discussion of their data, supplemented by others based on the parallaxes and proper motions of stars near the sun, now follows.

The quantities in the first and sixth columns of Table I (plotted as Curve 1 in Figure 6) give directly an approximation for the dependence of $\log \underline{T}_0$ on $\overline{M}$. For various reasons, however, the result is of low weight. First, the points at $M=4.1$ and 5.8 are seriously discordant, probably because of the inclusion of a considerable number of stars belonging to the high-speed group whose motions are not representative of the stars at large. These points were disregarded in drawing the curve, which thus becomes uncertain for the fainter magnitudes, the more so since the last two points depend on a small number of stars.

Further, the change in $\log \underline{T}_0$ with $\overline{M}$ is affected by a selection in the data which favors large proper motions. For small values of M the influence is probably negligible, since the intrinsically bright stars were among those selected on the basis of *apparent* magnitude, and there is no reason to suppose that either large or small motions were specially favored. But for stars intrinsically faint, the values of $\log \underline{T}_0$ are undoubtedly too large, as such stars are apparently faint and usually escape detection unless their motions are large.

Finally, the relation between log T_o and $\overline{M}$ is affected by the systematic error to which attention has been called by Eddington and Miss Douglas.[1] This is difficult to evaluate, but in general tends to neutralize the influence of selection in proper motions.

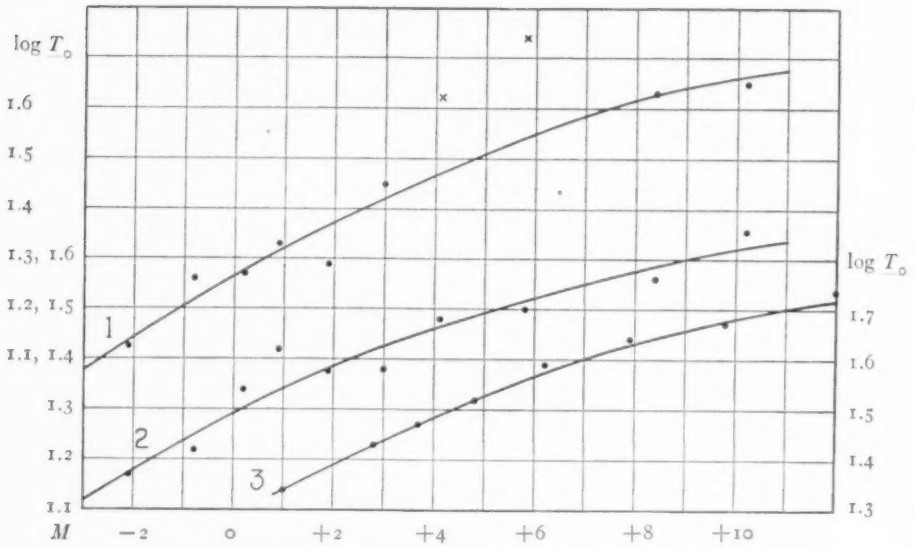


FIG. 6.—Progression with absolute magnitude in the logarithm of the geometrical mean tangential velocity, uncorrected for solar motion (T_o). Curve 1, from proper motions and spectroscopic parallaxes. Curve 2, from radial velocities of the same stars (data of Adams, Strömberg, and Joy, *Contribution* No. 210). Curve 3, from proper motions and parallaxes of 134 stars within about 12 parsecs.

Theoretically at least, the mean, $\bar{x}'$, of a group of observed values of any characteristic distributed according to the frequency function $f(x')$ is affected by a systematic error having the form[2]

$$\bar{x}-\bar{x}'=\overline{\epsilon^2}\frac{f'(x')}{f(x')} \tag{68}$$

where $\bar{x}$ is the true mean and $\overline{\epsilon^2}$ the mean-square error of x'. The logarithmic derivative of $f(x')$ is large when the dispersion in x' is small. If the dispersion is small, or the error of measurement relatively large, the systematic error of the mean may be serious.

[1] *Monthly Notices*, **83**, 112, 1923.

[2] See "A Troublesome Systematic Error," *Monthly Notices*, **84**, 15, 1923.

Further, if the relation of x to M is required, there is an added complication, since observation gives $\bar{x}'=G_1(\overline{M}')$ instead of the relation $\bar{x}=G(\overline{M})$, where M' is also systematically affected; and the trouble also naturally extends to any investigation of the change in $\bar{x}$ with $\overline{M}$.

For the tangential velocities under discussion, x represents $\log T_0$, and the systematic error is (errors in proper motion and apparent magnitude are disregarded)

$$\log \underline{T_0}-\log \underline{T_0'}=-0.2\ (\overline{M}-\overline{M}') \tag{69}$$

which is thus a simple function of the error in $\overline{M}'$ itself. The latter error also has the form (68), $f(x')$ now representing the frequency function of M'.

The weights in the last column of Table I, which are approximately proportional to the numbers of stars, indicate roughly the form of $f(M')$. Except at the ends, the curve is rather flat, although the maxima corresponding to giant and dwarf stars are clearly indicated. Since by (68) the systematic error depends on the slope of the frequency curve, it should not in this case be serious, especially since a proper combination of data on opposite sides of the maxima will partially neutralize the influence of a steep slope at the ends.

The average $\log \underline{T_0}$ for all the giant stars is therefore little affected. Further, the variation of $\log \underline{T_0}$ within the group, which undoubtedly is affected, is more or less controlled by the fact that the curve must represent the data for the faint stars. It is at least suggestive that a curve drawn on the basis of the data for the giants alone would be flatter than that adopted, which agrees with the direction in which the error operates.

Although less satisfactory than could be wished, the adopted curve for $\log \underline{T_0}$ is probably a good approximation as far as the third or fourth magnitude. This part is based on 875 stars, little if any affected by selection in proper motion, and the influence of the systematic error just discussed can scarcely be important. For comparison with other results, the ordinates of Curve 1 are given in the second column of Table VI.

An independent and more reliable determination of the change in log $\underline{T}_0$ with $\overline{M}$ can be made with the radial velocities by using the statistical relations connecting T_0 with θ. Were Maxwell's law strictly applicable to the space-velocities, results thus found would be entirely free from the influence of selection in proper motion. Actually there is some influence, but it must be small compared with that for the case of tangential velocities derived directly from proper motions and parallaxes.

A further advantage of using radial velocities is that the systematic error (68) affecting $\bar{\theta}_0$ is negligible, because $\bar{\epsilon}^2$ in this case is relatively small. The errors in $\overline{M}'$, which enter into the relation $\bar{\theta}_0 = G_1(\overline{M}')$, are still operative, however; but these behave as in the case of log T_0. Whatever the effect of systematic error on the progression in log $\underline{T}_0$ with M, its influence on the progression in $\bar{\theta}_0$ must be less.

The data for $\bar{\theta}$ in the eighth column of Table I are corrected for the sun's motion. The formulae giving the corresponding geometrical mean tangential velocities, *uncorrected* for solar motion, are (11), (12), (13), (26) and $\overline{T} = \pi\bar{\theta}/2$ from (14), and the numerical values from (46) and (67). The last two, by (11) to (14), give

$$\overline{T_0^2} = [0.484]\underline{T_0^2} \qquad \overline{T}^2 = [0.224]\overline{T}^2 = [0.224]\left(\frac{\pi}{2}\bar{\theta}\right)^2 \tag{70}$$

where the bracketed numbers are logarithms. Hence by (27)

$$[0.484]\underline{T_0^2} = [0.616]\bar{\theta}^2 + \tfrac{2}{3}V^2 .$$

The value of V used to correct the observed radial velocities was 21.48 km/sec.[1] Hence log $2V^2/3 = 2.488$, and finally

$$\underline{T_0^2} = 1.355\,\bar{\theta}^2 + 100.9 . \tag{71}$$

Had $\underline{\theta}$ instead of $\bar{\theta}$ been given, $\underline{T}$, and hence $\overline{T}$, could have been calculated by (23).

The results from (71) are in the last column but one of Table I. The values of log $\underline{T}_0$ are plotted as Curve 2, Figure 6, and the ordinates of this curve are in the third column of Table VI.

[1] *Mt. Wilson Contr.*, No. 210, p. 4; *Astrophysical Journal*, **54**, 13, 1921.

To strengthen the determination, values of log $\underline{T}_0$ were also calculated directly from the proper motions and parallaxes of stars of known distance having $\pi \geq 0''.079$. Luyten's list of stars of large proper motion[1] was the principal source, although some of the parallaxes of this list were revised. About 20 stars with proper motions less than Luyten's limit were taken from the spectroscopic list of 1646 stars,[2] and perhaps a few others within the limiting distance might have been found elsewhere, but, as it stands, the total of 134 stars must include practically all those known. Binaries were counted as single stars having a magnitude corresponding to the combined brightness. The values of log $\underline{T}_0$ and the corresponding $\overline{M}$ are in Table V. The ordinates of the resulting curve, No. 3, Figure 6, are in the fourth column of Table VI.

TABLE V

TANGENTIAL VELOCITY FROM μ AND π OF STARS NEAR THE SUN

M	log $\underline{T}_0$	No. Stars	$\overline{M}$	log $\underline{T}_0$	No. Stars
1.0	1.34	5	7.9	1.64	20
2.8	1.43	8	9.8	1.67	20
3.7	1.47	11	12.2	1.74	9
4.8	1.52	24			
6.2	1.59	37	Sum		134

TABLE VI

COLLECTED RESULTS FOR LOG T_0 (KM/SEC.)

M	Curve 1	Curve 2	Curve 3	Adopted
− 3	1.08	1.12		1.10
2	.14	.18		1.16
− 1	.21	.24		1.22
0	.27	.29		1.27
+ 1	.32	.34	1.33	1.32
2	.37	.39	.38	1.37
3	.42	.43	.43	1.41
4	.47	.46	.48	1.45
5	.51	.49	.52	1.48
6	.55	.52	.56	1.51
7	.59	.55	.60	1.54
8	.62	.58	.63	1.56
9	.64	.60	.66	1.58
10	.66	.62	.68	1.60
11	.68	.64	.70	1.61
+12			.72	1.62

[1] *Lick Observatory Bulletins*, **11**, 1 (No. 344), 1922.

[2] *Mt. Wilson Contr.*, No. 199; *Astrophysical Journal*, **53**, 13, 1921.

These data were derived by the same method as those represented by Curve 1, but are based on different though not wholly independent material. Since the parallaxes involved in Curve 3 are large, the individual values of log T_0 are more accurate than those underlying the first curve, and to a considerable extent this compensates for the much smaller number of stars. Another consequence of the large parallaxes is a more representative selection of the fainter magnitudes, although the dominating influence of large motions must still be active for large values of M, for the 134 stars used are only one-third the probable total[1] of those within the chosen limit of distance. The fact that Curves 1 and 3 are practically identical must not be taken too seriously. That they agree to $M=3$ or 4 is satisfactory and to be expected. because to this limit both sets of data should be practically free from the influence of selection; but, in view of the low weight of Curve 1 for large values of M, its coincidence with Curve 3 for these values must be largely accidental. In brief, therefore, the supplementary data from proper motions and parallaxes confirm those for the bright stars from *Contribution* No. 210; but the values of log T_0 for intrinsically faint stars, like those from Curve 1, are almost certainly too large.

The results from radial velocities represented by Curve 2 strengthen this conclusion; the progression in log T_0 is less pronounced, as would be expected. Whether the curve approximates the true progression is still somewhat in doubt, because the result depends upon the assumed constancy of the moduli of the distribution functions for log v and log T_0. Nevertheless, the change in log T_0 calculated from the radial velocities is much more probable than that found directly from the tangential velocities themselves. The zero point, on the other hand, is more reliably determined by the data of Curves 1 and 3 for the brighter stars.

The adopted relation between M and T_0 given in the last column of Table VI corresponds to the quadratic formula (73), and is practically that of Curve 2 corrected by -0.02 to bring the results into agreement with those derived from the proper motions for

[1] Based on the luminosity and density functions of Kapteyn and van Rhijn.

the interval $M=-1$ to $+3$. The values of $\log \overline{T}_0$ may be found by adding 0.12 to those in the last column of the table.

The final results for the distribution function of the uncorrected linear tangential velocities of the stars as a whole are

$$P(\log T_0)d(\log T_0)=\frac{h}{\sqrt{\pi}}e^{-h^2(\log T_0-\log \underline{T}_0)^2}d(\log T_0) \tag{72}$$

$$h=2.18, \qquad \log \underline{T}_0=1.27+0.052M-0.0019M^2 \tag{73}$$

$$\log \overline{T}_0-\log \underline{T}_0=0.121 \tag{67}$$

where $\underline{T}_0$ is expressed in kilometers per second and M is referred to the international zero point. To obtain velocities in astronomical units per year the constant term is to be replaced by 0.59.

In conclusion, it is desirable to indicate the accuracy with which the general relations of Section 3 are satisfied. These formulae depend upon the assumption of random distribution of the directions of the space-velocities, and have been used repeatedly throughout the discussion. For v and θ this could have been tested before, but since the data for T_0 in Table I include the solar velocity, the test for these velocities had to be deferred until the component of the sun's motion could be eliminated. With formulae analogous to (70) and (71) this is now possible.

On the basis of the data in Table I smooth curves were drawn for $\log \bar{v}$, $\log \bar{\theta}$, and $\log T_0$ as functions of M. Values of the ordinates for two-magnitude intervals are in the second, third, and fourth columns of Table VII. Values of $\log \overline{T}$ calculated from $\log \overline{T}_0$ by (12), (27), (46), and (67) are in the fifth column. Were the assumption of random distribution fully justified and were there no errors of observation or other uncertainties, the differences $\log \bar{v}/\bar{\theta}$ and $\log \bar{v}/\overline{T}$ in the last two columns, by (14), would be constant and equal, respectively, to 0.301 and 0.104. For the first series the condition is very nearly satisfied; for the second, the agreement is less satisfactory. The curve for $\log \overline{T}_0$, however, is relatively uncertain; further, $\log \overline{T}$ has been calculated on the assumption that the moduli are independent of M. Since the observed values of $\log \overline{T}/\underline{T}$ and $\log \overline{T}_0/\underline{T}_0$ are not constant, this necessarily introduces a progression into the values of $\log \bar{v}/\overline{T}$, presumably the influence of

selection in the proper motions. In the mean, however, the second series, as well as the first, satisfies the condition of random distribution very closely. As already mentioned, this does not mean that the directions of the space-velocities really are random, but merely that the integrated effect of systematic motions for all classes of stars scattered over the whole sky is negligible. For the present discussion this is the essential requirement.

TABLE VII

TEST OF RANDOM DISTRIBUTION OF v

M	$\log \bar{v}$	$\log \theta$	$\log T_*$	$\log \bar{T}$	$\log \bar{v}/\theta$	$\log \bar{v}/T$
− 2.....	1.34	1.03	1.27	1.18	0.31	0.16
0.....	1.48	1.17	1.41	1.36	0.31	0.12
+ 2.....	1.59	1.28	1.50	1.48	0.31	0.11
4.....	1.67	1.37	1.58	1.58	0.30	0.09
6.....	1.74	1.43	1.64	1.65	0.31	0.09
8.....	1.79	1.50	1.69	1.70	0.29	0.09
+10.....	1.84	1.55	1.72	1.76	0.29	0.08
Means					0.303	0.106

MOUNT WILSON OBSERVATORY
November 1923

NOTE ON THE SPECTROSCOPIC PARALLAXES OF A-TYPE STARS

By BERTIL LINDBLAD

ABSTRACT

Comparison of the Harvard and Mount Wilson classifications of A-type stars.—The difference between the reduction curves for deriving absolute magnitude used by Adams and Joy for A-type stars with "sharp" and "nebulous" spectral lines respectively seems to be due entirely to a systematic difference in the Mount Wilson type for these two kinds of spectra. Within the Harvard interval A0–3 the Mount Wilson spectral type seems to be correlated with the exposure-ratio for the regions λλ 3895–3907 and λλ 3907–3925, measured in short objective-prism spectra. There are strong indications that the subdivisions by Adams and Joy of the A-type in their relation to the original Harvard sequence, at least for the Harvard interval A0–3, have the character of a *luminosity classification;* i.e., a classification dependent on the atmospheric density-gradient of a star in addition to the surface temperature. This circumstance helps to explain the probably small range of luminosity within each Mount Wilson subdivision.

Improvements in the method of classifying short and narrow objective-prism spectra of early type are suggested.

Adams and Joy have made an important contribution to the classification of stellar spectra in their work on the A-type stars[1] (more recently extended to stars of type B),[2] with the aim of obtaining criteria of absolute luminosity. Their classification is claimed to be a refinement of the usual Harvard classification, but still essentially equivalent to it. It is the object of this note to discuss the relation between the Mount Wilson and Harvard classifications of the A-type stars, with some reference to the writer's experience with a classification of short objective-prism spectra of this type.[3]

The Mount Wilson observers do not use the intensities of the H and K lines relative to the hydrogen lines, which form the most important criterion in the Harvard classification; their classification is based instead on the strength of the metallic arc lines. Moreover, they separate stars with sharp and nebulous lines, and derive a curve for the relation between absolute magnitude and spectral type for each one of these two classes separately.

[1] *Mount Wilson Contributions*, No. 244; *Astrophysical Journal*, **56**, 242, 1922.

[2] *Ibid.*, No. 262; *ibid.*, **57**, 294, 1923.

[3] *Ibid.*, No. 228; *ibid.*, **55**, 85, 1922.

Table I gives the mean of the Mount Wilson type, for the "sharp" and "nebulous" series respectively, corresponding to the successive Harvard subdivisions for the stars of known trigonometric or group parallax in Adams' and Joy's list, which were used by those authors for obtaining the reduction curves mentioned. Most of the stars have been classified in *Harvard Annals* **28** and **56.** It is evident that there is a systematic difference between the "nebulous" and "sharp" subdivisions in the Mount Wilson system for a fixed subdivision in the Harvard system later than B9. The mean difference "sharp"—"nebulous" is nearly +0.3, if the whole A-type is taken as a unit. This difference corresponds almost exactly to the displacement from the "nebulous" to the "sharp" curve in Figure 1 of Adams' and Joy's paper. The vertical difference between

TABLE I

Harvard Spectrum	Mount Wilson "Sharp"	No.	Mount Wilson "Nebulous"	No.	"Sharp"—"Nebulous"
B8.....	B8.5	2	B8	3	+0.05
B9.....	B9	1	B9	3	0.0
A0.....	A2.3	18	A0.5	21	+0.18
A2.....	A5.3	8	A1.3	16	0.40
A3.....	A5.8	9	A4.1	7	0.17
A5.....	A8.5	2	A3.7	17	0.48
F0.....	A7.2	5	A5.5	20	+0.17
F2.....	.....	.....	A7	3	.....
F5.....	F1.5	2	A8	1	+0.35

these curves is thus certainly not a direct effect of luminosity, but merely a reflection of a systematic difference in classification between "nebulous" and "sharp" spectra. The difference in question is well shown also if the stars are grouped according to color (e.g., from Hertzsprung's table)[1] instead of according to Harvard type. In what follows I therefore add 0.3 to the "nebulous" types to reduce them to the system of "sharp" types.

It is evident from Table I that in the interval A0–3 of the Harvard system the Mount Wilson subdivisions give a more detailed classification of the spectra than the original Harvard classification. As the exposure-ratio for the regions λλ 3895–3907 A and λλ 3907–3925 A measured by me, and found to vary with the absolute

[1] *Annalen van de Sterrewacht te Leiden,* Deel **14,** Eerste Stuk, 1922.

magnitude within this interval of type, is certainly dependent to a considerable extent on the intensities of some arc lines of *Si* and *Fe* in the region λλ 3895–3907, and as the Mount Wilson classification is based on the increased strength of the metallic arc lines, there should be a relation between the Mount Wilson classification and the measured exposure-ratio. It is certainly not correct to connect the variation in the exposure-ratio with the phenomenon sharp—nebulous lines, as is done by Adams and Joy.[1] It is probably true that the exposure-ratio in question is dependent also on the intensity of the "wings" of the hydrogen line $H\zeta$, but there does not seem to be any definite connection between the sharp or nebulous appearance of the metallic lines and the extension of the wings of the hydrogen lines. There are some reasons for believing, however, that there exists a correlation between the strength of the arc lines and the extension of the hydrogen wings.

TABLE II

Spectrum M.W.	log *E*	Spectrum H.	No.
A2.9.....	0.16	A0.8	13
A8.7.....	0.22	A2.2	6

It is clear that a comparison of the Mount Wilson type and the exposure-ratio can be made only for one general strength of the hydrogen lines at a time; i.e., for a limited interval of the Harvard type. A direct comparison, including the results for a few stars of the Hyades, Praesepe, and Ursa Major groups of Harvard type A0–3, is given in Table II.

It seems further, in agreement with what has been said above, that the Mount Wilson classification in its relation to the Harvard sequence, especially within A0–3, has much of the character of a luminosity classification, analogous to a classification according to the intensities of the characteristic enhanced and arc lines of *Sr*, *Ca*, etc., in the late types. The increased density-gradient at the surface of the low-luminosity stars may be thought to favor the appearance of the arc lines, as well as the intensity and extension of the wings of the hydrogen lines.

[1] *Loc. cit.*, p. 263.

Figure 1 gives the relation between Mount Wilson type and absolute magnitude (according to the parallaxes in Adams' and Joy's table)[1] for the Harvard types A0-3. A0 stars are represented by circles, A2 stars by points, and A3 stars by crosses. The curve drawn is Adams' and Joy's reduction curve for the "sharp" group. The absolutely faint stars of Harvard A0–3 are evidently placed in late subdivisions of the Mount Wilson sequence, where their lumi-

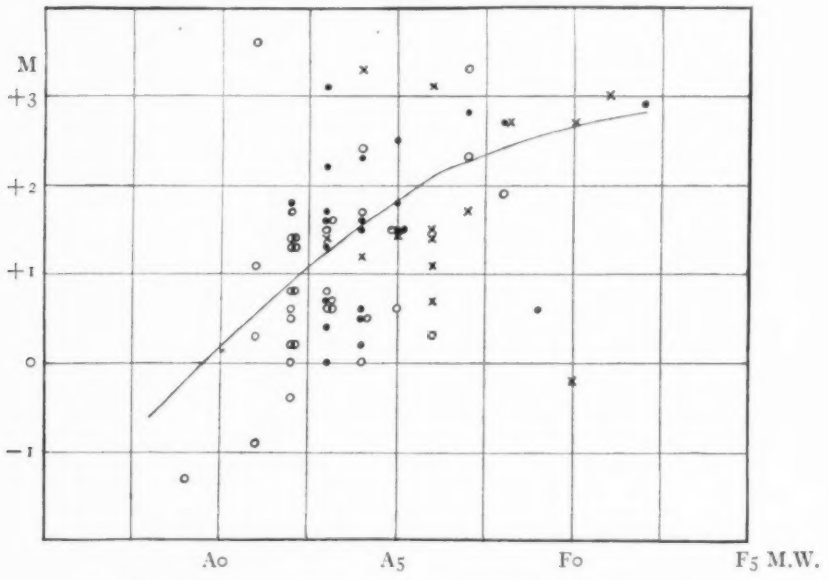


FIG. 1.—Relation between Mount Wilson type and absolute magnitude for stars of early Harvard A-type. o, Harvard A0; •, A2; x, A3.

nosity is more in agreement with the mean luminosity of the subdivision. It is clear that this effect will reduce the dispersion of the luminosity within one and the same type. This means, on the other hand, that the Harvard and Mount Wilson systems have very little in common for the A- and early F-type stars.

I wish to add here that the measurement of the exposure-ratio for the small ultra-violet regions mentioned is rather difficult and

[1] Only the group parallaxes of the Taurus, Ursa Major, and Praesepe stars have been used, and in these cases the trigonometric parallaxes were omitted. For Praesepe the parallax due to Kohlschütter, 0".0072, was assumed instead of Rasmuson's value.

subject to serious disturbances in the case of objective-prism spectra of low dispersion. In later work these measurements have therefore been replaced by a division of the spectra of early A-type into six groups following the general extension of the wings of the hydrogen lines and, within the "winged" groups, an increase in color shown in the general intensity-distribution of the photographic spectrum and the appearance of a slight characteristic break at λ 3907 due to the increased strength of the arc lines. At the same time the spectral sub-class in the Harvard sequence is estimated by observing the intensity of the K line and the general strength of the hydrogen lines.

Some probably faint stars like the Hyades star, *Groningen* No. 87, seem to be somewhat peculiar in this classification. The K line is faint, but the hydrogen lines rather weak for an early A-type; it would be easy to make the mistake of thinking them only very slightly winged, which would put the star into a far too high luminosity-class. The general "redness" of the spectrum is, however, easily discovered upon closer inspection, and the star ought no doubt to be placed in the A5- or F-type in spite of the faint K line. In the case of 87 Hyades the Mount Wilson type is F2; the type estimated at Harvard is A2.

I am obliged to Mr. Yngve Öhman for valuable help in preparing this note.

Astronomiska Observatorium, Upsala
September 1923

THE FORM OF THE LUMINOSITY FUNCTION[1]

By FREDERICK H. SEARES

ABSTRACT

Relation of luminosity function to mean-parallax formula.—The uncertainty in the distribution of luminosity for the intrinsically faint stars can be traced to an uncertainty in their mean parallaxes. Grouped according to two characteristics, m and μ, the data are so meager that the three coefficients in the mean-parallax formula cannot be accurately determined. The empirical formula of Kapteyn and van Rhijn can be written, however, as a linear function of $H = m + 5 \log \mu$, containing but two constants. The use of such a formula, involving a grouping according to the single characteristic H, would avoid much of the difficulty.

A theoretical formula for mean parallax can be derived from the distribution functions for density, tangential velocity, and luminosity, the first two of which are known with little uncertainty. When the error curve of Kapteyn and van Rhijn is adopted for the distribution of absolute magnitudes, the theoretical formula for mean parallax differs from the empirical relation only in the presence of a small term in m (or μ), which for stars near the sun is too small to be detected. Hence, if the luminosity function is really an error curve, the mean parallaxes of the stars of low luminosity (those known are near the sun) should be represented by a linear function of H having coefficients agreeing with those of the formula of Kapteyn and van Rhijn.

For an actual test it is convenient to determine b and d from

$$\overline{M} - H = b + dH ,$$

which is the equivalent of the theoretical formula. The results from about 500 stars, all with proper motions $> 0''.3$, and hence near the sun, show that a linear relation holds to $H = 7.5 (\overline{M} = 6.5)$, with coefficients agreeing with those of Kapteyn and van Rhijn. For fainter stars the relation is also approximately linear, but the values of b and d are different [Fig. 1; equations (46), (47)]. Hence the mean parallaxes of intrinsically faint stars are not represented by the formula of Kapteyn and van Rhijn (Table IV); they are almost independent of m, and hence nearly proportional to μ.

Form of the luminosity function.—The disagreement thus found must be attributed to the Gaussian curve adopted for the distribution of absolute magnitudes. For stars brighter than about $M = 7$ the function of Kapteyn and van Rhijn is applicable. For the fainter stars another curve is required, roughly that shown in Figure 2. From the data available it is impossible to tell whether the curve ultimately descends or not.

The density function.—The frequencies of the absolute magnitudes seem to increase continuously to the limit now known. Hence the number of intrinsically faint stars, near the sun at least, must be much greater than that corresponding to the error-curve function. To $M = 15$ the total density per cubic parsec appears to be about twice that hitherto adopted for all absolute magnitudes. The relative densities up to about a thousand parsecs (Table VI) are in close agreement with those of Kapteyn and van Rhijn, and for an interval of 1.6 in $\log \rho$ can be closely represented by a quadratic exponential of the usual form. The coefficients depend on the limiting values of $\log \rho$, and are quite different for near and distant stars (Table VII).

The luminosity function expresses the relative numbers of stars of different absolute magnitudes in any limited region of space. Were the distances of all stars known, the frequencies could be found by simple grouping and counting. Actually, but few parallaxes have been measured; the stars of known distance are not a

[1] *Contributions from the Mount Wilson Observatory*, No. 273.

representative sample and their absolute magnitudes do not give directly the true frequencies for any definite region.

With rare exceptions, stars intrinsically faint are also faint as seen from the earth; and, unless their proper motions are large, usually remain hidden among the multitudes of other apparently faint stars. The luminous stars, on the other hand, attract attention because they are bright, and are likely to be observed even if their motions are small. The known stars within a given region thus present a selection favoring large motions, which becomes more and more pronounced with increasing absolute magnitude.

To determine the luminosity function we must, in effect, find how many stars of each luminosity remain undiscovered. Whatever the method, it must utilize, or at least be independent of, the selective characteristics of the data. The usual procedure is based on the mean-parallax formula, and in principle is unobjectionable, because this relation *presupposes* a grouping of the data according to proper motion (and apparent magnitude). There are, however, serious practical difficulties, and the discussion in *Mount Wilson Contribution*, No. 271, shows that the descending branch of the function is still in doubt.

It is not difficult to trace this uncertainty to the mean parallaxes of the intrinsically faint stars, or more remotely, to the fact that so few stars of low luminosity are known. Were such objects available in large numbers, there would be no difficulty in finding their mean parallaxes, and hence no large uncertainty in the luminosity function. As it is, the data are so meager, that when grouped according to *two* characteristics, m and μ, it is difficult to determine accurately the constants in a formula of the usual type.

A simplified procedure is suggested by the fact that the mean parallax formula of Kapteyn and van Rhijn[1]

$$\log \pi = -0.690 - 0.0713\, m + 0.645 \log \mu \qquad (1)$$

can be written in the form

$$\log \pi - \log \mu = -0.690 - 0.0710\, H - 0.0003 m\,, \qquad (2)$$

a linear function of H, where

$$H = m + 5 \log \mu\,. \qquad (3)$$

[1] Equation (8) *Mt. Wilson Contr.*, No. 188; *Astrophysical Journal*, **52**, 23, 1920.

The term in m has no significance. Its coefficient may not actually be as small as indicated; but the precision of the constants of (1) indicated by the discussion in *Contribution* No. 188 is such that the right member of (2) can be regarded as practically independent of m.[1]

Further, for stars of constant m

$$\overline{M} = m + 5 + 5 \log \overline{\pi} . \tag{4}$$

The combination of (2), (3), and (4) gives

$$\overline{M} = +1.55 + 0.645\, H - 5\delta , \tag{5}$$

where

$$\delta = \log \overline{\pi} - \log \underline{\pi}$$

is a small constant whose value is known. Hence the mean absolute magnitude of the stars of magnitude m and motion μ is also a linear function of H.

Equations (2) and (5) involve two unknown constants and the single characteristic H. The result is empirical, but justifiable from general considerations for certain special conditions; and, when applicable, affords a simple means of dealing with observational data which meets no serious obstacle in the scattering results for stars of low luminosity. Grouped according to H, for example, the observed values of M determine b and c in

$$\overline{M} = b + cH. \tag{5a}$$

By reversing the transformation just made, we then have for the *geometrical* mean parallax:[2]

$$\log \overline{\pi} - \log \mu = (0.2b - 1) + 0.2(c - 1)H. \tag{2a}$$

[1] In this connection, see Luyten, *Lick Observatory Bulletin*, **11**, 39 (No. 345), 1922. For convenience I have omitted the constant term 5 included by Luyten in his definition of H.

[2] The left-hand member of this expression, as it stands, is the logarithm of the geometrical mean tangential velocity of the stars of magnitude m and motion μ. Hertzsprung, *Astronomische Nachrichten*, **208**, 89, 1919, has made use of similar formulae involving two constants.

It will be shown later that (1) does not represent the distances of the low-luminosity stars. Hence it is not safe to assume that equations of the form of (5*a*) and (2*a*) hold for these objects. Moreover, it is not at present a question of finding an improved mean-parallax formula for the subsequent improvement of the luminosity function, but rather one of testing the form of a function already derived.

It is preferable, therefore, to begin with the luminosity function of Kapteyn and van Rhijn, which was based on equation (1), and derive analytically the formula for mean parallax. This should lead back to equation (1), but in such a way as to show whether the formula corresponding to a luminosity law of the error-curve type is really independent of an explicit m for the stars of low luminosity. If so, (5*a*) can then be used to test the consistency of the data for these stars with the function of Kapteyn and van Rhijn.

The analytical derivation of the mean-parallax formula also involves the density law and the frequency function for linear tangential velocity. The density law is itself a consequence of the luminosity function and observational data (star counts) which are open to no great uncertainty. Further, the distribution function for velocity has been established by the discussion in *Contribution* No. 272 with sufficient accuracy for the present purpose. Any disagreement with observation appearing as a result of the foregoing comparison must therefore be attributable to the luminosity function itself.

ANALYTICAL DERIVATION OF MEAN-PARALLAX FORMULA

The three fundamental distribution functions are,

(1) *Luminosity:*

$$\left.\begin{aligned} \phi(M)dM &= dM \cdot e^{p+qM+rM^2} \\ q=+0.4278, \quad r&=-0.07944, \quad M=m+5\log\pi \end{aligned}\right\} \tag{6}$$

(2) *Density:*

$$\left.\begin{aligned} \Delta(\rho) &= e^{h+k(\log\rho)+l(\log\rho)^2} \\ k=+5.481 \quad l&=-1.508 \end{aligned}\right\} \tag{7}$$

Relations (6) and (7) are from the investigation by Kapteyn and van Rhijn in *Contribution* No. 229.[1] Numerical values of p and h are not required; the values of k and l are for galactic latitude 30°, but represent satisfactorily the average density for all latitudes.[2] The value of $\Delta(\rho)$ is the number of stars per cubic parsec at the distance ρ.

(3) *Linear tangential velocity*, uncorrected for the sun's motion:

$$\left.\begin{aligned} &P(\tau)d\tau = \sqrt{\frac{-\omega}{\pi}} \cdot e^{\omega(\tau-\bar{\tau})^2} d\tau \\ &\tau = \log T_0 \qquad \omega = -h^2 = -4.75 \\ &\bar{\tau} = \log \bar{T}_0 = +0.80 + 0.033\, M - 0.0019\, M^2 \end{aligned}\right\} \tag{8}$$

These relations are based on results given in *Contribution* No. 272, in which it is shown that a Gaussian distribution for τ is consistent with a Gaussian distribution of the logarithms of the space-velocities, a well-established observational result by Adams, Strömberg, and Joy. To agree with (6), the constants in the expression for $\bar{\tau}$ are referred to Kapteyn's zero-point for M (+5, international scale), and presuppose

$$T_0 = \frac{\mu}{\pi} = \mu\rho\,, \tag{9}$$

in other words, that the tangential velocities are expressed in astronomical units per year. Further,

$$M = m - 5 \log \rho \tag{10}$$

where ρ is expressed in parsecs.

Expressions for τ^2 and $0.4M\tau$ as functions of M will be needed later. In order that a necessary integration may be effected, these must not be of higher degree than the second. The data in Table VI, *Contribution* No. 272, give (Kapteyn's scale)

$$\tau^2 = 0.64 + 0.049\, M - 0.0016\, M^2 \tag{11}$$

$$0.4M\tau = +0.294\, M + 0.0136\, M^2\,. \tag{12}$$

[1] *Astrophysical Journal*, 55, 242, 1922.

[2] *Mt. Wilson Contr.*, No. 271; *Astrophysical Journal*, 59, 11, 1924.

Representation by (11) is within 0.01; by (12), within 0.05 or 0.06, or about 6 per cent for the large deviations. The rather unsatisfactory agreement for (12) has very little influence, however, on the final result.

The mean-parallax formula has been derived by Schwarzschild[1] from functions of the form of (6), (7), and (8) for the cases in which τ is a constant and a linear function of M. What follows is an extension of his investigation in that it assumes $\bar{\tau}$, except in the case of (12), to be a quadratic in M. The results, moreover, are expressed in terms of the characteristic H.

The number of stars having absolute magnitudes between M and $M+dM$ and logarithms of velocities between τ and $\tau+d\tau$, situated in a shell of radius ρ and thickness $d\rho$, is

$$4\pi\rho^2 d\rho\Delta(\rho)\phi(M)dMP(\tau)d\tau\,. \tag{13}$$

By (9) and (10)

$$\phi(M)dM=\phi(m-5\log\rho)dm$$

$$P(\tau)d\tau=P(\log\mu+\log\rho)\frac{\text{Mod.}\,d\mu}{\mu},$$

which connect the functions for M and τ with frequency functions for m and μ, referred to the intervals dm and $d\mu$. Since all of the variable factors in (13) can be expressed in terms of the characteristic

$$x=\log\pi=-\log\rho \tag{14}$$

(13) may be written

$$N(x)dx=K\Delta(\rho)\phi(m-5\log\rho)P(\log\mu+\log\rho)\rho^2 d\rho \tag{15}$$

where[2]

$$K=\frac{4\pi\ \text{Mod.}}{\mu}\,dmd\mu \tag{16}$$

and where $N(x)dx$ indicates the number of stars having apparent magnitudes between m and $m+dm$ and proper motions between μ and $\mu+d\mu$ for which x lies between x and $x+dx$. All the factors

[1] *Astronomische Nachrichten*, **190**, 361, 1912; **198**, 217, 1914.

[2] The factor π is here of course 3.1416, and not to be confused with the parallax.

in (15) except K involve exponentials of the first or second degree in x. The equation therefore has the form

$$N(x)dx = K_1 e^{\alpha+\beta x+\gamma x^2}dx \tag{17}$$

where the numerical values of the coefficients are such that γ is negative.

Expressions for $\bar{x}$ and $\bar{\pi}$ can be found by the usual method of calculating a mean, but more directly by noting that (17) represents a Gaussian distribution of x about the mean value $-\beta/2\gamma$. Hence

$$\bar{x} = \log \bar{\pi} = -\frac{\beta}{2\gamma}. \tag{18}$$

Further, from the general property of a Gaussian distribution of logarithms

$$\log \bar{\pi} - \log \underline{\pi} = \frac{1}{4\kappa^2 \text{ Mod.}} \tag{19}$$

where κ is the modulus of the Gaussian function. Since $\kappa^2 = -\gamma$,

$$\log \bar{\pi} = \log \underline{\pi} - \frac{a'}{4\gamma} \qquad a' = 1/\text{Mod.} \tag{20}$$

Equations (18) and (20) are independent of K_1, which includes K, and hence independent of the intervals $d\mu$ and dm.

Since (18) and (20) involve only β and γ, terms in x and x^2 in the exponentials of (15) are all that need be considered. From $x = -\log \rho$, $\rho^2 d\rho = -a'e^{-3a'x}\,dx$, which contributes $-3a'x$. By (7), $\Delta(\rho)$ yields $-kx+lx^2$, and from (6) we find that $\phi(m-5\log\rho)$ gives $(5q+10rm)x+25rx^2$. It is convenient to segregate the terms arising from $P(\tau)$. Hence for the foregoing partial results write

$$\beta_1 = -3a' - k + 5q + 10rm \tag{21}$$

$$\gamma_1 = l + 25r\,. \tag{22}$$

The expression of $P(\tau)$ as a quadratic exponential in x presents the chief difficulty, and unfortunately is possible only with the aid of the interpolation formulae (11) and (12). This obscures the effect of the coefficients of $\bar{\tau}$, which, as Schwarzschild has shown, is a matter of considerable interest.[1]

[1] *Astronomische Nachrichten*, **198**, 217, 1914.

By (3), (9), and (10)

$$\tau = 0.2(H-M)\,. \tag{23}$$

The exponent of $P(\tau)$ can therefore be written

$$\omega(\tau-\bar{\tau})^2 = \omega\{0.04(H-M)^2 - 0.4H\bar{\tau} + 0.4M\bar{\tau} + \bar{\tau}^2\}$$

which by (8), (11), and (12) reduces to a quadratic in M containing the terms ωBM and ωCM^2 where

$$B = 0.343 - 0.0932\,H \tag{24}$$

$$C = 0.0520 + 0.00076\,H. \tag{25}$$

The resulting terms in x and x^2 are

$$\beta_2 = 5\omega(B+2Cm) \tag{26}$$

$$\gamma_2 = 25\omega C. \tag{27}$$

Hence the complete expressions for β and γ are

$$\beta = -3a' - k + 5q + 10rm + 5\omega(B+2Cm) \tag{28}$$

$$\gamma = l + 25r + 25\omega C. \tag{29}$$

For comparison with the empirical formula (2) we require the difference

$$\log \bar{\pi} - \log \mu = \frac{-\beta - 0.5a' - 2\gamma \log \mu}{2\gamma}\,. \tag{30}$$

With the aid of (3) this readily reduces to

$$\log \bar{\pi} - \log \mu = \frac{\delta' - 0.4\gamma H - 5\omega B + 0.4lm}{2\gamma} \tag{31}$$

where

$$\delta' = 2.5a' + k - 5q. \tag{32}$$

The equation for $\log \pi - \log \mu$ is the same as (31) except that the coefficient of a' is to be replaced by 3.0.

The expressions for B and C from (24) and (25) can now be introduced, and likewise numerical values of r, l, and δ'. The value

for ω derived in *Contribution* No. 272 is also available [see (8)], but it is desirable to retain this constant for the present in order to see how small changes in its value affect the mean parallax. We thus find for (31)

$$\log \bar{\pi} - \log \mu = \frac{+9.097 - 1.715\omega + 1.398H - 0.054\omega H - 0.0076\omega H^2 - 0.60m}{-6.99 + 2.60\omega + 0.038\omega H}. \quad (33)$$

Since ω is of the order of -5 and H differs little from the absolute magnitude, the quadratic terms in H have little influence. As far as this characteristic is concerned, the expression is very nearly linear. It is not, however, independent of m, and in this respect differs from (2). The coefficient of m is only 0.03, but in a range of ten magnitudes changes the mean parallax in a two-to-one ratio.

A reference to (31) shows that this term arises from the dispersion coefficient l in the distribution function for density. Within about 60 parsecs of the sun, however, the density is nearly constant,[1] and for this region both k and l must be small. The equation for mean parallax appears therefore to be different for near and distant stars. For the latter, if at all, (33) should reduce to the formula of Kapteyn and van Rhijn. For $k = l = 0$, we find

$$\log \bar{\pi}_0 - \log \mu = \frac{+3.616 - 1.715\omega + 0.794H - 0.054\omega H - 0.0076\omega H^2}{-3.972 + 2.60\omega + 0.038\omega H}. \quad (34)$$

This expression is conveniently used to test the behavior of ω. The value of ω corresponding to the modulus of the distribution function for tangential velocity found in *Contribution* No. 272, directly from observed data, is -4.75. Values of $\log \bar{\pi}_0 - \log \mu$ computed from (34) for a series of values of H with $\omega = -4.75$ are in the second column of Table I. The residuals resulting from a comparison with (2) are in the fourth column. Two other series of residuals, for $\omega = -4.50$ and -5.00, are in the third and fifth columns, respectively.

It will be shown later that equation (2) does not represent the data of observation for values of H greater than about 8. The

[1] *Mt. Wilson Contr.*, No. 188, Table VI; *Astrophysical Journal*, **52**, 23, 1920.

large residuals for $H=15$ can therefore be left out of consideration. As for the others, there is little choice between the three series, whence it appears that the mean parallaxes are not very sensitive to small changes in ω. The representation for $\omega=-5.00$, corresponding to $h=2.24$, is slightly better than that for $\omega=-4.75$; but equation (2) itself is affected by some uncertainty, and there is at present no valid reason for changing the value of the modulus ($h=2.18$) found in *Contribution* No. 272. In view of a possible

TABLE I

COMPARISON OF ANALYTICAL AND EMPIRICAL FORMULAE—CONSTANT DENSITY

H	Eq. (34) $\omega=-4.75$	Eq. (34)−Eq. (2)			Eq. (34) −Eq. (35)	Eq. (34) −Eq. (36)	$\log \bar{\pi}_0 - \log \mu$	$\log \bar{\pi}_0 - \log \bar{\pi}_e$
		-4.50	-4.75	-5.00				
0.....	−0.721	−0.033	−0.031	−0.028	−0.002	−0.01	−0.791	+0.070
5.....	1.040	0.006	+0.005	+0.015	+0.001	+0.02	1.107	0.067
10.....	1.428	0.045	−0.028	−0.011	−0.002	−0.01	1.492	0.064
15.....	−1.874	−0.142	−0.119	−0.097	+0.001	−0.10	−1.934	+0.060

influence of selection and the fact that the modulus has been assumed to be independent of absolute magnitude, the agreement is close. That (34) can be brought into substantial agreement with (2) for any value of ω is in itself a satisfactory result, for (34), it must be remembered, has been derived from the distribution functions for luminosity, density, and stellar velocity, while (2) is an empirical formula based on observational data.

Without loss of accuracy, (34) can be written in the form (π_0 indicates $k=l=0$)

$$\log \bar{\pi}_0 - \log \mu = -0.72 - 0.0580H - 0.00127H^2 . \qquad (35)$$

The differences between (34) and (35) are given in the sixth column of Table I. For a range of ten units in H a satisfactory substitute for (35) is the simple linear relation

$$\text{(Analytical)} \qquad \log \bar{\pi}_0 - \log \mu = -0.71 - 0.071H \qquad (36)$$

$$\text{(Kapteyn and van Rhijn)} \quad \log \bar{\pi} - \log \mu = -0.69 - 0.071H . \qquad (2)$$

The agreement of (36) with (34) is shown by the differences in the seventh column of the table. Equation (2) is repeated for comparison. The analytical formula for the case of constant density is therefore practically the equivalent of the empirical formula of Kapteyn and van Rhijn.

The relations for geometrical mean parallax corresponding to (35) and (36) will be needed later, and may be introduced at this point. The only change necessary is the use of 3.0 for the coefficient of a' in (32). The first term in the numerator of (34) thus becomes $+4.768$, and with $\omega = -4.75$, as before, the values of $\log \pi_0 - \log \mu$ are shown in the eighth column of Table I. These are represented by

$$\log \pi_0 - \log \mu = -0.79 - 0.0575H - 0.00126H^2 \qquad (37)$$

and

$$\log \pi_0 - \log \mu = -0.78 - 0.070H \qquad (38)$$

with residuals similar to those in the sixth and seventh columns of the table.

The numbers in the second and eighth columns of Table I lead to the differences between the arithmetical and geometrical mean parallaxes given in the last column. These determine the modulus κ of the distribution function for $\log \pi$ (for stars of a given m and μ), which, as remarked in connection with (17), is a Gaussian error-curve. From (19) we derive

for $H =$	0	5	10	15
$\kappa =$	2.87	2.93	3.00	3.10
$r =$	0.166	0.163	0.159	0.154

(39)

where $r = 0.477/\kappa$ is the probable deviation.

We now return to the consideration of (33), which henceforth must be considered as applying only to distant stars. Using again $\omega = -4.75$ and proceeding as in the case of (34), we find that (33) can be replaced by a quadratic in H, or, approximately, by a linear relation, as follows:

$$\log \bar{\pi} - \log \mu = -0.890 - 0.0783H - 0.00094H^2 + 0.030m \qquad (40)$$

$$\log \bar{\pi} - \log \mu = -0.890 - 0.0877H + 0.030m. \qquad (41)$$

The numerical values of (33) (disregarding the term in m) are in the second column of Table II; the representation by (40) and (41), in the third and fourth columns. To obtain the formula for the geometrical mean parallax, $+10.249$ must be substituted for the first term in the numerator of (33). The formulae involving $\log \pi$ which correspond to (40) and (41) are not required, but can easily be found from the numerical values in the fifth column of Table II. The differences $\log \bar{\pi} - \log \pi$ and the corresponding values of κ and r are in the remaining columns.

It will be noted that with the three fundamental distribution functions as a theoretical basis, the modulus of the frequency

TABLE II

COMPARISON OF FORMULAE—DENSITY VARIABLE

H	Eq. (33)*	Eq. (33) −Eq. (40)	Eq. (33) −Eq. (41)	$\log \pi$ $-\log \mu$*	$\log \bar{\pi}$ $-\log \pi$	κ	r
0.....	−0.892	0.000	0.00	−0.951	+0.059	3.12	0.153
5.....	1.305	0.000	+0.01	1.362	0.057	3.18	0.150
10.....	1.769	+0.002	0.00	1.824	0.055	3.24	0.147
15.....	−2.276	0.000	−0.07	−2.328	+0.052	3.33	0.143

* The term in m is omitted.

function for $\log \pi$ depends on H and differs slightly for near and distant stars. The value of r used by Kapteyn and van Rhijn[1] in deriving the luminosity function was 0.19. Since this was based on observational data for stars having large parallaxes, it is to be compared with the values given in (39).

The chief results of the analytical development of the mean-parallax formula are expressed by (35) and (36), applying to stars near the sun in the region of nearly constant density, and (40) and (41), which refer to more distant objects. The only difference in form is the presence of the term in m in the equations for the distant stars. Unless the dispersion coefficient l in the expression for density is zero, or so small as to be negligible, this term must appear.[2] Since the coefficient of m in equation (1) was based upon the

[1] *Mt. Wilson Contr.*, No. 188, p. 7; *Astrophysical Journal*, **52**, 23, 1920; *Groningen Publication*, No. 8, p. 25.

[2] The term in m can of course be replaced by one in μ. See equation (3).

parallactic motions of stars as faint as the eleventh magnitude,[1] it is surprising to find so little trace of it in equation (2). The most extreme of the several values of the coefficients of (1) given in *Contribution* No. 188 will not account for more than a tenth of the theoretical requirement.

Some light is thrown upon the matter by an examination of the mean parallaxes recently published by van Rhijn.[2] His formulae are not easily compared with (40), and it is simpler to deal with his tabular results and try to represent their deviations from (1), by a relation of the form

$$VR-Eq.(1)=\Delta \log \pi=a+bH+cm\,. \qquad (42)$$

For a limited range in proper motion this is quite possible. In selecting values for the comparison it is important to avoid very small values of the proper motion. The formulae developed here, like the equation of Kapteyn, have the defect that for $\mu=0$ they give $\pi=0$.[3] Van Rhijn's formulae are intended to meet this difficulty, and hence for small values of μ are not comparable with the relation above. Further, mean parallaxes larger than $0''.015$ have not been used, in order to restrict the comparison to a region such that the adopted dispersion coefficient of the density function comes fully into operation.

The differences in the logarithms $VR-Eq.$ (1) were plotted for different values of μ with H as abscissae. A glance at the curves shows that a linear term in H is sufficient to represent the differences. For $\mu=0''.02$ to $0''.20$ the curves are approximately parallel, but do not coincide. Hence c is not zero. In general the differences in this region are well represented by

$$\Delta \log \pi=-0.18-0.015\,H+0.026\,m\,. \qquad (43)$$

The coefficient of m decreases with increasing μ and apparently becomes zero at about $\mu=0''.2$.

[1] *Mt. Wilson Contr.*, No. 188, p. 5; *Astrophysical Journal*, **52**, 23, 1920.

[2] *Groningen Publication*, No. 34, p. 37, and Table 19, 1923.

[3] The theoretical development does not cover the case of very small proper motions because the distribution function for τ is not applicable to velocities less than 10 km/sec. See *Contribution*, No. 272.

The addition of (43) to (2), which is the equivalent of (1), gives for the mean parallaxes of van Rhijn's tables in the region examined:

(van Rhijn) $\log \bar{\pi} - \log \mu = -0.87 - 0.086H + 0.026m$ (44)

(Analytical) $\log \pi - \log \mu = -0.89 - 0.088H + 0.030m$. (41)

The analytical formula (41), repeated for comparison, is practically identical with this result.

The comparison with van Rhijn's tables can also be extended to the case of the nearer stars. For $\mu = 0''.2$ to $0''.5$ the differences, although not zero, are nearly independent of m, and can be represented by a formula which agrees with (36), except that the coefficient of H is 0.063 instead of 0.071. For still larger motions the agreement of the tables with (2) and (36) is bad. See Table IV.

The inference from these results is that the nearer stars have played a dominating influence in the determination of the coefficients in the formula of Kapteyn and van Rhijn. We shall see later that the densities within about 200 parsecs can be closely represented by a quadratic exponential in $\log \rho$ with a value of l about one-sixth that usually adopted. The m-term for stars within this distance would be $0.005\ m$, which is too small to be detected from the available data.

The discussion thus far has established the form of the equation for mean parallax which corresponds to the adopted functions for luminosity, density, and velocity. The relation is so nearly linear in H that much observational material would be required to detect the departure and determine the coefficient of the quadratic term. The origin of this term is obscured by the use of interpolation formulae for $\bar{\tau}$, $\bar{\tau}^2$, etc., but it will be seen later that it arises from the quadratic term in the expression for $\bar{\tau}$. Further, the term in m can enter only in the manner indicated, and must be small for stars near the sun. Finally, it may be remarked that the analytic expressions for mean parallax agree, not only in form, but also numerically with the empirical formulae derived from observational data.

MEAN PARALLAXES OF STARS NEAR THE SUN

This clears the way for an examination of the mean parallaxes of the nearer stars by the method outlined in connection with

equations (2*a*) and (5*a*). The use of these relations, which are independent of m except as it enters through H, is now seen to be legitimate, because of the slow change in the density near the sun. Among these nearer stars we find all the known objects having luminosities so low that they fall on the descending branch of the curve. If the luminosity function is really a Gaussian error-curve, (5*a*) applied to these stars must yield values of b and c which, substituted into (2*a*), will reproduce the coefficients of (2).

It will be noted that the grouping according to H automatically takes care of the selection of stars which lie on the descending branch of the curve. Since the maximum of the curve is at $M=7.7$, we should expect from (5) that such stars would be characterized by values of H greater than about 10.

For numerical reasons it is convenient to use (5*a*) in the form

$$\bar{M}-H=b+dH. \qquad (44a)$$

The corresponding expression for the parallax is then

$$\log \pi-\log \mu=(0.2b-1)+0.2dH. \qquad (44b)$$

The observational data used to determine the coefficients b and d were drawn from two sources—Luyten's list of stars of large proper motion[1] and the catalogue of 1646 spectroscopic parallaxes.[2] The former includes all known stars having $\mu \geq 0''.5$, and gives H directly,[3] and also M for the stars of known distance. The values of M have been very generally revised in order to bring them into agreement with the weighted mean of all reliable determinations of the parallax. Three stars, Nos. 5, 233, and 265, were rejected because of uncertainty. Binaries were counted as single stars with a magnitude equivalent to the combined brightness of the components. With these restrictions 325 stars are available, which, with few exceptions, are within 50 parsecs of the sun, and hence within the region of nearly constant density.

[1] *Lick Observatory Bulletin*, **11**, 1 (No. 344), 1923.

[2] *Mt. Wilson Contr.*, No. 199; *Astrophysical Journal*, **53**, 13, 1921.

[3] The value tabulated by Luyten is $H=m+5+\log \mu$, which is 5 units greater than the H used in this paper.

The individual values of $M-H$ were arranged in order of H and formed into means for groups of 10 or 11 stars each. The range in M for a given H is considerable, and, to reduce the effect of accidental deviations to a minimum, adjacent groups were combined wherever possible without unduly widening the interval for H. Although the values of M differ greatly in precision, they were given equal weights in forming the means. For the present general survey this is sufficient and avoids the use of a complicated system of weighting. The final means are collected in the first half of Table III and plotted as points in Figure 1.

TABLE III

OBSERVATIONAL DATA FOR $M-H$

$\mu \geq 0''.5$			$0''.3 \geq \mu < 0''.5$		
H	$\bar{M}-H$	No. Stars	H	$\bar{M}-H$	No. Stars
0.89	+1.30	10	1.00	+0.92	10
3.08	0.31	20	2.01	+0.65	10
4.49	+0.07	30	2.61*	−0.21	30
5.19	−0.18	30	3.26	+0.10	30
5.90	0.85	30	4.11	+0.18	30
6.38	0.65	30	4.97	−0.09	20
6.78	1.19	30	6.04	0.39	20
7.20	0.85	32	6.58	0.70	20
7.72	1.11	33	7.10	−0.86	11
8.28*	1.52	30			
9.20	1.13	20	Total	.....	181
10.56	1.08	20			
13.15	−1.30	10			
Total	.....	325			

* Revision in text.

The data taken from the spectroscopic list include 181 stars having $0''.3 \geq \mu < 0''.5$. The parallaxes of 9 stars showing some uncertainty were combined with trigonometric values; but in general the values of M were used without modification. The maximum distance is about 80 parsecs so that these stars are also within the region of practically constant density. The treatment of the data was the same as for those from Luyten's list. The means are given in the second part of Table III and plotted as circles in Figure 1. The figure shows at once that the relation of $\bar{M}-H$ to

H is not linear, although linear formulae apparently can be used to represent the points on either side of the value $H = 7.5$.

As a preliminary to the derivation of these formulae, certain mean values require comment. The discordant point at $H = 2.61$ (indicated by a cross) includes three subgroups of 10 stars each. Nineteen of the 20 stars in the first two of these have tangential

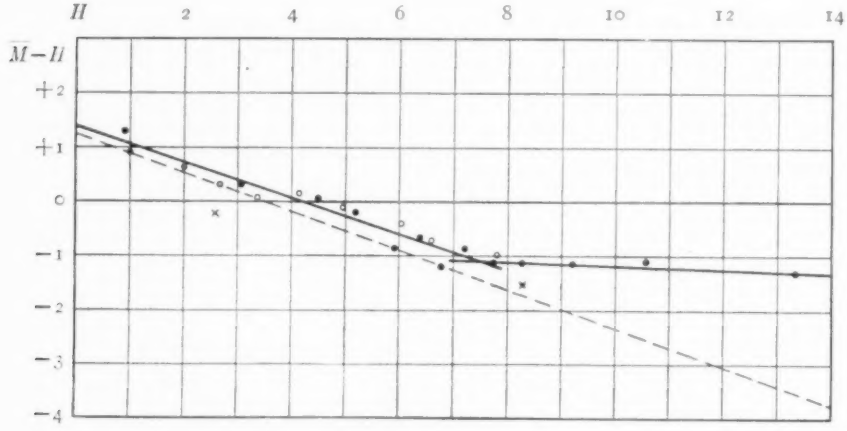


FIG. 1.—Relation between mean absolute magnitude and $H = m + 5 \log \mu$, an equivalent of the formula for the geometrical mean parallax (in logarithmic form) of stars of given m and μ. The dotted line corresponds to the formula of Kapteyn and van Rhijn, and is in agreement with the theoretical relation resulting from a luminosity function of the error-curve type and a Gaussian distribution of the logarithms of the tangential velocities, in a region where the stellar density is a linear function of $\log \rho$ ($l = 0$). The circles represent 181 stars from the list of spectroscopic parallaxes, $\mu = 0''.3$ to $0''.5$; the points, 325 stars from Luyten's list, $\mu > 0''.5$. The formula of Kapteyn and van Rhijn represents these data to $M = 6.5$ (systematic difference, about 3 per cent in π). The wide departure for intrinsically faint stars indicates that the frequencies of the absolute magnitudes of these stars are not distributed in accordance with an error-curve. (The last circle is incorrectly plotted.)

velocities larger than the mean for stars of corresponding absolute magnitude, and the average velocity for the 20 stars is twice the mean for stars at large having the same luminosity. Nearly all these stars are of types F and G, with absolute magnitudes between +2 and +3, and apparently belong to the special group of high-speed stars. The number of these objects is much in excess of the percentage that would be found in a group composed of a representa-

tive selection of all the stars having the same value of H. The first two subgroups were accordingly rejected; the third subgroup gives

$$H=2.79 \qquad M-H=+0.33 \qquad (10 \text{ stars}).$$

The points at $H=5.90$ and 6.78 (30 stars each) show large deviations; but the individual values of $M-H$ present nothing that admits of any change. Several of the parallaxes are small and uncertain, but at present there is no basis for revision. On the other hand, four of the 20 stars included in the mean for $H=8.28$ (indicated by a cross) have very large values of $M-H$, namely, -4.0. -4.1, -4.5, and -4.7. The rejection of these stars is probably justifiable, since their mean tangential velocity is 370 km/sec. The revised mean is

$$H=8.26 \qquad M-H=-1.10 \qquad (26 \text{ stars}).$$

With these revisions, solutions for the coefficients of (44*a*) gave

$$M-H=+1.40-0.331H \qquad (373 \text{ stars}, H<7.5) \qquad (44c)$$

$$M-H=-0.84-0.029H \qquad (76 \text{ stars}, H>8.0). \qquad (45)$$

Because of their critical position at the turning-point of the curve, the 33 stars included in the point at $H=7.72$ were not used in either solution. The corresponding expressions for the geometrical mean parallax are by (44*b*),

$$\log \pi-\log \mu=-0.72-0.066H \ (H<7.5) \qquad (46)$$

$$\log \pi-\log \mu=-1.17-0.006H \ (H>8.0). \qquad (47)$$

The probable errors of the coefficients of (46) are ± 0.01 and ± 0.0025, respectively. In the case of (47) the uncertainty is much greater; the coefficient of H, for example, may lie anywhere between estimated limits of $+0.020$ and -0.020.

The formula of Kapteyn and van Rhijn, equation (2), modified to give $\log \pi$ is

$$\log \pi-\log \mu=-0.75-0.071H. \qquad (48)$$

Equation (38), derived analytically on the assumption $k=l=0$, is

$$\log \pi_0-\log \mu=-0.78-0.070H. \qquad (38)$$

Since (46) depends on only 373 stars, all with $\mu \geqq 0''.3$, its agreement with these results is remarkably close. Equation (47), on the other hand, even with all reasonable allowance for uncertainty, deviates widely from all the other formulae. Considered collectively, the data for the 482 stars can in no sense be regarded as defining a linear relation, nor can it be assumed that the observed deviations from such a relation are to be explained as the result of neglected terms of higher order in H. Thus reference to (37), which includes the term in H^2, indicates that the departure from the simple linear relation is opposite to that shown by Figure 1.

The lines represented by (46) and (47) intersect at $H = 7.5$, which corresponds to $M = +6.4$. The point of intersection is not sharply determined, but apparently the value of M does not differ greatly from the adopted maximum of the luminosity curve ($M_0 = +7.7$). This circumstance and the general agreement of (46) with (38) and (48) seem to mean that the luminosity function of Kapteyn and van Rhijn is applicable as long as we are dealing with stars whose magnitudes fall on the ascending branch of their curve. The available data for the stars of lower luminosity are represented by the linear relation (47); this may correspond to a Gaussian distribution of the absolute magnitudes, but if so, the constants must differ from those of the function of Kapteyn and van Rhijn.

As a whole, therefore, the luminosity function seems not to be a symmetrical curve. As far as the analytical relations are concerned, the frequencies of the absolute magnitudes may even increase indefinitely with increasing M, for the important point in arriving at the expressions above for $\log \pi$ is the Gaussian form of (17). This will occur if γ is negative; but (29) shows that a negative γ does not necessarily mean that r is negative, which is the condition to be satisfied if the frequencies for M are ultimately to decrease.

DETAILED RESULTS FOR STARS OF LOW LUMINOSITY

Since the conclusion as to the asymmetrical form of the luminosity curve turns upon the behavior of about 75 stars, it is desirable to give at least part of the evidence in detail. The necessary data for the 50 stars included in the last three points of Figure 1 are given in Table IV. With the exception of several revised parallaxes,

TABLE IV

COMPARISON OF LOW-LUMINOSITY STARS WITH THE FORMULA OF KAPTEYN AND VAN RHIJN AND THE TABLES OF VAN RHIJN

L. O. Bulletin, No. 344		m	μ	H	π OBSERVED	LOG $\pi_{obs.}$ −LOG $\bar{\pi}$	$\bar{\pi}$ VAN RHIJN
No.	Designation						
283	Be. C 1500	9.5	0".634	8.7	0".064	+0.37	0".035
738	Cbr. E 14250	8.9	0.91	8.7	0.020	−0.29	.049
174	Be. A 1866	9.0	0.93	8.8	0.015	−0.42	.049
639	Pots. Astr. 1214, 608	9.7	0.671	8.8	0.089	+0.50	.035
645	61 Cygni	5.2*	5.267	8.8	0.310	+0.15	.30
704	B.D.+1°4637	10.3	0.55	9.0	0.004	−0.75	.028
411	Oxf. Astr.+25°86067	10.4	0.54	9.1	0.033	0.18	.028
455	Cbr. E 7086	9.2	0.961	9.1	0.080	0.32	.049
581	Lpz. II 9034	9.5	0.812	9.1	0.048	0.17	.042
698	Hels. 13170	9.4*	0.87	9.1	0.256	+0.87	.044
240	Be. C 1255	8.5*	1.390	9.2	0.102	+0.27	.068
385	Lund 5723	9.5*	0.884	9.2	0.073	0.32	.044
185	B.D.+17°1320	9.5	0.88	9.4	0.090	0.43	.044
652	Cordoba 29191	6.65	3.532	9.4	0.238	0.25	.180
740	Alb. 8164	8.7	1.393	9.4	0.183	+0.54	.068
658	Harv. II 7623	9.4	1.052	9.5	0.051	+0.11	.050
425	Be. B 5072	8.9*	1.39	9.6	0.070	0.13	.065
514	W. Ott 5811	9.1*	1.234	9.6	0.150	0.52	.058
46	Hels. 914	8.7	1.547	9.7	0.090	+0.20	.072
213	Lei. 3313	8.2	1.962	9.7	0.050	−0.15	.093
231	B.D.+67°552	9.3	1.101	9.7	0.114	+0.45	.053
526	B.D.+42°2810	9.6	1.061	9.7	0.069	0.25	.049
542		9.1	1.39	9.8	0.134	0.43	.062
749		10.2	0.856	9.9	0.101	0.52	.039
548	Chri. 2710	9.5	1.334	10.1	0.223	+0.69	.057
573	Hels. 9945	8.3*	2.307	10.1	0.278	+0.55	.102
578	B.D.+5°3993	9.7	1.247	10.2	0.086	0.31	.053
11	Bo. 187	8.0*	2.890	10.3	0.282	+0.47	.124
133	Lei. 1521	8.6	2.200	10.3	0.045	−0.21	.093
399	Be. A 5003	8.5	2.298	10.3	0.216	+0.45	.096
158	Strb. 1611	8.8	2.222	10.5	0.157	+0.34	.091
328	Gr. 1830	6.46	7.047	10.7	0.104	−0.32	.....
701	B.D.+53°2911	9.5	1.482	10.7	0.039	−0.07	.062
82	Near W Androm.	12.4	0.5	10.9	0.051	+0.53	.019
287	22 H Camelopardalis	7.6	4.778	11.0	0.409	+0.46	.183
398	Be. A 4999	9.6	1.86	11.0	0.099	+0.26	.070
497		10.7	1.232	11.2	0.133	0.58	.045
304	Chri. 1745	9.2	2.986	11.6	0.215	0.43	.103
713	Cordoba 31353	7.4	6.900	11.6	0.282	+0.18	.....
458	Wash. 5584	8.9*	3.682	11.7	0.029	−0.53	0.123

* Magnitude corresponding to total light of system.

TABLE IV—*Continued*

L. O. Bulletin, No. 344		m	μ	H	π Observed	log $\pi_{obs.}$ −log π	π van Rhijn
No.	Designation						
729	Wolf 1039	11	1.44	12.0	0.059	+0.21	0″.047
288	Bo. 7899	8.9	4.519	12.2	0.181	0.22	.141
747	Cordoba 32416	8.3	6.112	12.2	0.191	0.11	.....
582	17 Lyrae C	11.3	1.658	12.4	0.121	0.49	.049
37	Wolf 33	11.2	1.58	12.6	0.064	0.25	.049
262	Near W Ursae Majoris	14.1	0.924	12.8	0.094	0.66	.022
152	Kapteyn's Star	9.2	8.764	13.9	0.309	0.28	.....
298	.....	12.5	2.718	13.9	0.337	0.83	.054
35	van Maanen's Star	12.3	3.01	14.7	0.255	0.72	0.060
556	Barnard's Star	9.67	10.252	14.8	0.520	+0.51	.....

* Magnitude corresponding to total light of system.

these are from Luyten's list, the stars being arranged in the order of increasing H.

The significant fact is the systematic character of the differences $\log \pi - \log \pi$ given in the seventh column of the table, only eight of which are negative. These differences are the deviations of the logarithms of the measured parallaxes in the sixth column from $\log \pi$ calculated by (48), which is nothing but equation (2), the formula of Kapteyn and van Rhijn, modified to give the geometrical mean. The value of $\log \bar{\pi} - \log \pi$ used for this reduction is +0.06, from the last column of Table I, and depends upon the dispersion coefficients r, l, and ω in (6), (7), (8). The coefficient l was assumed to be zero in deriving the data in Table I, and for the stars in question the value of r is in doubt; the most important coefficient, however, is ω, which seems to be well determined. The adopted value of the constant for reduction to geometrical mean parallax was checked by a direct calculation from the measured parallaxes of the 50 stars, which were divided for the purpose into several groups extending over small intervals in m and μ. The mean thus found, including errors of observation, is +0.09, but the rejection of one discordant and very uncertain parallax reduces the result to +0.07.

The arithmetical means of the differences in the seventh column of Table IV for three groups of 20, 20, and 10 stars each, corresponding to the last three points of the curve, are

$$+0.19, \quad +0.29, \quad +0.43 . \tag{49}$$

On the average, therefore, the observed parallaxes for these 50 stars are about twice the geometrical mean corresponding to the formula of Kapteyn and van Rhijn.[1] Theoretically, these differences are affected by a small systematic error in that they involve the mean logarithm of an observed quantity π' whose observation errors presumably have a Gaussian distribution. At least this is the usual assumption with regard to the errors affecting trigonometric parallaxes, of which the present list is chiefly composed. It can be shown, however, that the systematic error arising from this source is negligible.

The differences (49) indicate clearly the departure from the formula of Kapteyn and van Rhijn. To show that they are accounted for by the adopted formula (47), we may form the difference (47) *minus* (48), which gives

$$-0.42+0.065\,H\,.$$

For the mean values of H corresponding to the three groups of stars, namely, 9.20, 10.56, and 13.15, this expression reduces to

$$+0.18 \qquad +0.27 \qquad +0.43\,. \tag{50}$$

The differences between (49) and (50)

$$+0.01 \qquad +0.02 \qquad 0.00$$

represent the residuals O.−C. in $\log \pi$ calculated by (47) for the last three points of the curve.

It seems clear therefore that the formula of Kapteyn and van Rhijn does not represent the parallaxes of the stars of low luminosity. Nor are they represented by the tables of van Rhijn. This is seen at once by comparing the observed parallaxes in the sixth column of Table IV with the values in the last column, which were interpolated from Table 19 of *Groningen Publication*, No. 34. For only seven of the 45 stars falling within the limits of the tables is the mean parallax in excess of the observed value. The systematic deviation is in the same direction and of nearly the same amount as in the case of the formula of Kapteyn and van Rhijn.

[1] A difference of this order has been noted by Russell for about a dozen K5 and M stars of low luminosity. See *Astrophysical Journal*, **26**, 150, 1910.

The small coefficient of H in (47) implies that the mean parallax of stars of low luminosity is nearly independent of m and, practically speaking, is determined by μ alone. As an average ($H=11$)

$$\pi=\frac{1}{17}\mu \qquad \bar{\pi}=\frac{1}{15}\mu . \tag{51}$$

APPROXIMATE FORM OF LUMINOSITY FUNCTION

To gain an idea of the form of the luminosity function corresponding to (46) and (47), consider (31) for the relatively simple case in which $\bar{\tau}$ is a linear function of M:

$$\tau=s+tM. \tag{52}$$

Then by (23)

$$\tau-\bar{\tau}=0.2(H-M)-s-tM$$
$$(\tau-\bar{\tau})^2=A+BM+CM^2$$

where

$$B=0.4\epsilon(s-0.2H), \qquad C=0.04\epsilon^2, \qquad \epsilon=1+5t. \tag{53}$$

With these values of B and C, (31) reduces (using the form for $\log \bar{\pi}$) to

$$\log \bar{\pi}-\log \mu=\frac{\delta-2\omega\epsilon s-0.4(\gamma-\omega\epsilon)H+0.4lm}{2\gamma} \tag{54}$$

where

$$\delta=3a'+k-5q, \qquad \gamma=l+25r+\omega\epsilon^2 . \tag{55}$$

It will be noted that the quadratic term in H has now disappeared. For constant density, $k=l=0$, and

$$\log \bar{\pi}_0-\log \mu=k_1-k_2H \tag{56}$$

where

$$k_1=\frac{3a'-5q-2\omega\epsilon s}{50r+2\omega\epsilon^2}, \quad k_2=\frac{5r+\omega\epsilon t}{25r+\omega\epsilon^2} . \tag{57}$$

With s, t, and ω known, these relations can be used to determine q and r, the essential constants of the luminosity function. Using the round value -5 for ω, and introducing $3a'=3/\text{Mod.}=6.908$, we have

$$k_1=\frac{1.382-q+2\epsilon s}{10r-2\epsilon^2} \qquad k_2=\frac{r-\epsilon t}{5r-\epsilon^2} . \tag{58}$$

From the adopted quadratic expression for $\bar{\tau}$,

$$t=\frac{d\bar{\tau}}{dM}=+0.033-0.0038M. \tag{59}$$

The average absolute magnitudes of the stars on the two sections of the curve in Figure 1 are about $+5$ and $+10$, or on Kapteyn's scale, 0 and $+5$, respectively. The corresponding values for t are $+0.033$ and $+0.014$. Hence

$$\bar{\tau}=+0.79+0.033M \quad (H<8) \tag{60}$$

$$\bar{\tau}=+0.85+0.014M \quad (H>8) \tag{61}$$

in which the constant terms are such that the formulae represent the average values of $\bar{\tau}$ for four or five magnitudes on either side of the foregoing values of M.

The coefficients of (46) and (60) substituted into (58) then give $r=-0.076$, $q=+0.71$. The adopted values are -0.079 and $+0.43$. The approximation for r is satisfactory, but that for q is bad. The trouble arises chiefly from the uncertainty in the constant term in (46), which is 0.06 smaller than that in the analytical formula (38). The form of the first of (58) is such that this difference appears in q multiplied about three and one-half times.

Using the coefficients of (47) and (61) and applying the same procedure to the second part of the curve in Figure 1, we find a *positive* value of r, which would mean that the luminosity curve corresponding to the present data does not descend at all. From about $H=8$ the frequencies of M continue to increase indefinitely.

In order that the curve may descend, r must be negative. From the second of (57)

$$5r(1-5k_2)=\omega\epsilon(\epsilon k_2-t)$$

where k_2 is a small quantity, ϵ is positive, and ω negative. The condition is therefore

$$k_2>\frac{t}{1+5t}, \tag{62}$$

a relation of considerable theoretical interest. Since $t=+0.014$, the existence of a descending branch to the luminosity curve would require $k_2>+0.013$. The observed value is $+0.006$, but the uncer-

tainty is several times the outstanding difference. Whether positive or negative, r is undoubtedly small. The coefficient q, however, is certainly positive. Hence the frequencies of M must continue to increase far beyond $H=8$; whether the curve ultimately descends or not cannot be determined from the present data.

Although the results can scarcely have any permanent value it has seemed worth while, as an illustration of the conclusions above, to calculate the luminosity function corresponding to the data of Figure 1. This has been done by the method described by Kapteyn in *Groningen Publication*, No. 11, equations (46) and (47) being used to derive the mean parallaxes. The statistical data for proper

TABLE V

APPROXIMATE LUMINOSITY FUNCTION

M*	log $\phi(M)$		Diff.	M	log $\phi(M)$		Diff.
	This Paper	K and τR†			This Paper	K and τR	
−0.64....	5.58−10	5.46−10	+0.12	+ 8.36...	7.86−10	7.84−10	+0.02
+0.36....	6.16	6.00	0.16	9.36...	7.88	7.76	0.12
1.36....	6.66	6.47	0.19	10.36...	7.92	7.61	0.31
2.36....	7.05	6.88	0.17	11.20...	8.06	7.39	0.67
3.36....	7.34	7.21	0.13	12.36...	8.11	7.10	1.01
4.36....	7.58	7.47	0.11	12.36...	8.11	6.75	1.36
5.36....	7.74	7.67	0.07	13.36...	8.11	6.3	+1.8
6.36....	7.84	7.80	0.04	+14.36...	8.13	.....	.....
+7.36....	7.87	7.85	+0.02				

*International scale. †*Contribution* No. 188, Table IV.

motions were taken from Table 19 of *Groningen Publication*, No. 30. The value used for the modulus of the distribution function for log π was $\kappa=3.00$ [(see 39)]. The results thus found are given in Table V and illustrated in Figure 2.

Since (46) differs but slightly from (48), that portion of the curve which depends upon (46) should agree closely with the curve of Kapteyn and van Rhijn. This in fact is the case, as may be seen from the differences in the fourth column of Table V and from Figure 2. The chief point of interest is the wide departure beyond $M=8$. It should be emphasized, however, that Figure 2 indicates little more than that a wide departure exists, for slight changes in the measured parallaxes would alter the course of the curve

considerably. But it is impossible that the uncertainties should be such as to account wholly for the divergence from the symmetrical curve of Kapteyn and van Rhijn. As shown in the preceding section, this would require us to suppose that the largest-known parallaxes are affected by a systematic error which is of the order of the true values of the parallaxes themselves. The large irregularity in the curve at $M=8$ can scarcely be real and probably represents a discontinuity in the distribution of the parallaxes introduced by the sudden transition from (46) to (47) at $H=7.5$.

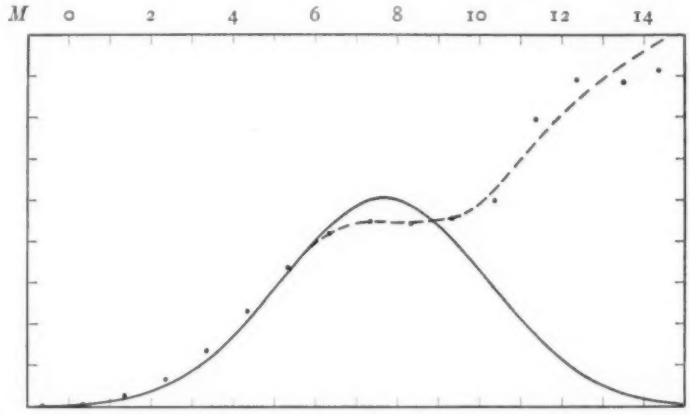


FIG. 2.—The error-curve indicates the luminosity function of Kapteyn and van Rhijn. The points represent the distribution of absolute magnitudes based on the stars of large proper motion used for Fig. 1. The sudden transition from one linear relation for M to the other (Fig. 1) is partly responsible, at least, for the peculiar distribution of frequencies near $M=8$.

In view of these results one need scarcely urge upon parallax observers the importance of additional observations of stars of low luminosity. The form of the luminosity function cannot be definitely determined until these are available in considerable numbers. The criterion for selection is simple: Observe every star of unknown or uncertain distance for which $H>7$. Strictly speaking, the limit for H should be somewhat smaller, for the transition from (46) to (47) must also be accurately determined. Luyten's list in *Lick Observatory Bulletin*, No. 344, is a convenient guide in the preparation of an observing program; only it must be remembered that Luyten's H is five units larger than that used here.

The interest of the question of course centers around its physical significance. The luminosity function of Kapteyn and van Rhijn implies that stars fainter than about the twentieth absolute magnitude are so infrequent as to be practically non-existent. Since mass is closely related to luminosity, this would mean the very infrequent occurrence of small masses, certainly of small luminous masses, and probably of all masses, either luminous or dark, below some limit. The frequency function for all masses in the universe, from the largest star to the smallest dust particle, would then show a maximum, corresponding more or less closely to the maximum of the luminosity curve, followed by a minimum for some relatively small mass and a subsequent increase in frequencies as still smaller masses (planets, asteroids, meteoric material) were considered. A curve such as that in Figure 2, on the other hand, suggests that the most frequent mass of the stellar order of size is less than that of any known luminous star, and raises the question whether there is a definitely marked minimum for any value of the mass.

Whether this is actually the case or not, it seems likely that stars of low luminosity are much more numerous than hitherto has been supposed. This is certainly true of the nearer objects, and since the relative frequencies of stars on the ascending branch of the curve of Kapteyn and van Rhijn seem to be independent of distance, it is perhaps not unreasonable to suppose that the excess of intrinsically faint stars exists everywhere. If so their total number must be enormous.

STELLAR DENSITY NEAR THE SUN

For the neighborhood of the sun Kapteyn and van Rhijn found a total density of 0.0451 stars per cubic parsec. With a luminosity function such as that shown in Figure 2, total density has no meaning; we can determine only the total number of stars per unit volume brighter than a given limit of absolute magnitude. Results for various limiting values of M are given in Table VI. Actually, these do not depend upon the luminosity curve of Figure 2, but represent rather an intermediate step in the determination of the curve itself, and are calculated from the numbers of proper motions, the dispersion in $\log \pi$ for a given m and μ, and the mean-parallax formulae (46) and (47). Since (46) is nearly the same as (1), the

total densities to successive limits of M to $M = 7$ or 8 resulting from it must agree with those found by Kapteyn and van Rhijn. This is really the case, as may be seen by comparing the second and third columns of Table VI which refer to nearly the same limiting M.

The missing densities for the lower limits of M can of course be supplied by assuming that the relative frequencies of M are the same at all distances; but the provisional character of the results scarcely justifies the calculation. As they stand, the data suggest that the total number of stars per cubic parsec brighter than $M = 15$

TABLE VI

NUMBER OF STARS PER CUBIC PARSEC BRIGHTER THAN M

log ρ	K and vR^* 7.69	Limiting M							
		7.86	8.86	9.86	10.86	11.86	12.86	13.86	14.86
0.9.....	0.023	0.028	0.035	0.042	0.050	0.060	0.073	0.087	0.098
1.1.....	.023	.026	.033	.040	.048	0.058	0.069	0.078	.....
1.3.....	.023	.024	.030	.035	0.041	.....	.....	.....	.....
1.5.....	.023	.023	0.028	0.033	.....	.....	.....	.....	.....
1.7.....	.023	.022	.....	.....	.....	.....	.....	.....	.....
1.9.....	.020	.020	.....	.....	.....	.....	.....	.....	.....
2.1.....	.016	.017	.....	.....	.....	.....	.....	.....	.....
2.3.....	.012	.014	.....	.....	.....	.....	.....	.....	.....
2.5.....	.009	.011	.....	.....	.....	.....	.....	.....	.....
2.7.....	.006	.008	.....	.....	.....	.....	.....	.....	.....
2.9.....	0.003	0.004	.....	.....	.....	.....	.....	.....	.....

* For galactic latitude 30°. From Table 1*b*, *Contribution* No. 229. The limit 7.69 for M corresponds to the maximum of the luminosity curve of Kapteyn and van Rhijn.

is about twice the total for all magnitudes found by Kapteyn and van Rhijn.

The fact that there is no indication of convergence toward a definite total density with increasing M does not prevent the calculation of the relative densities at different distances. The function representing these is determinate, provided the distribution of the frequencies of M is the same at all distances, and clearly more complicated than the quadratic exponential (7) usually adopted, which is highly advantageous because it permits a simple analytical treatment of many problems.

It is of interest to see how the coefficients h, k, and l behave, and what the limits of applicability of the quadratic exponential for $\Delta(\rho)$ are when fitted to the observed densities in different regions.

The maximum $\Delta(\rho)$ corresponds to $\log \rho = -k/2l$, and an approximate calculation for different positions of the maximum gives the results in the second to the fifth columns of Table VII. The values in the last column are those of Kapteyn and van Rhijn derived from the luminosity function and the data for star counts. The first two lines of the table give the limiting values of $\log \rho$ within which the representation of the relative densities is satisfactory—to 1 per cent everywhere except at the upper limit itself, where the deviations are a little larger. The range is about 1.6 in $\log \rho$, or from the lower limit of distance to about 40 times this limit. In dealing with the stars collectively, that is, all galactic latitudes together, we should expect on this basis that the quadratic formula, with the constants of Kapteyn and van Rhijn, would represent the

TABLE VII

COEFFICIENTS OF DENSITY FUNCTION
($\Delta(\rho) = 1$ at Sun)

Log ρ for max. $\Delta(\rho)$	0.3	0.7	1.1	1.5	1.8
Upper limit log ρ	1.9	2.3	2.7	2.9	.....
h	−0.021	−0.143	−0.613	−2.02	−5.19
k	+0.069	+0.327	+0.962	+2.42	+5.48
l	−0.115	−0.235	−0.438	−0.806	−1.51

densities from 60 parsecs up to 2000 or 3000 parsecs. For many purposes this is sufficient, but it presupposes that the nearer stars, among which are most of those of measured distance, have been disregarded. Any treatment of measured parallaxes by analytical methods therefore requires the use of other values of the constants. For distances up to 200 parsecs the second set in Table VII is applicable.

With these now available, the errors in the theoretical formula (36) [and (37) and (38) also] arising from the assumption $k = l = 0$ can be estimated. A simple calculation shows that the coefficients found above are scarcely affected, while the neglected term in m, as already mentioned, is $+0.005\ m$. For a mean magnitude of 6, this term would equal +0.03, and, combined with the constant term, would bring the theoretical formulae into almost exact agreement with that of Kapteyn and van Rhijn.

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RADIAL VELOCITIES OF STARS OF SPECTRAL CLASS R[1]

By R. F. SANFORD

ABSTRACT

Radial velocities.—Table I gives the radial velocities of twenty-nine stars of spectral class R, one of spectral class S (B.D.+6°3898), and a variable (B.D.+14°1598) that is peculiar and bears some resemblance to class S. B.D.+20°5071 is noteworthy in having probably the largest constant radial velocity so far found for an isolated star. In Table II the velocities are shown to be essentially free from systematic errors. Intercomparison of the radial velocities of R and N stars furnishes further justification for certain systematic corrections which Moore applied to his velocities for stars of class N, and connects the radial-velocity systems of classes R and N in a satisfactory manner.

Solar motion.—Although the data are meager, the solar motion referred to this group of stars seems to be in agreement with that derived from stars in general.

Absolute magnitude.—Parallactic and peculiar motions combined with radial velocities give a mean absolute magnitude of −1.5. Although the exact agreement with Moore's value for N stars is accidental, the two classes seem to be nearly alike in luminosity, a point to be borne in mind in any theory of stellar evolution. The line-intensity method of determining absolute magnitudes is at present inapplicable to class R stars because of lack of standards; but one such pair of lines shows a behavior similar to that observed among giant G stars. This is of interest, because good evidence for establishing the spectral sequence in the order G, R, N, has been given elsewhere.

Space motions.—Stars of class R display the same characteristics of space motion as stars in general. In particular, the two stars of largest radial velocity are strikingly akin to the high-speed stars of the other spectral classes.

Low-dispersion spectrograms.—Table IV gives for six stars of class R some observational data concerning radial velocity and spectrum as derived from spectrograms obtained with a low-dispersion focal-plane spectrograph. The velocities depending solely upon H and K of calcium are the only ones derived for four of the stars, and show merely the order of magnitude.

Use of red sensitive plates.—An intercomparison of spectrograms taken with Seed's Gilt Edge 30 and ammoniated Ilford Special Panchromatic plates gives an idea of the relative exposure times necessary for plates of proper density for classes R5, R8, and Nb. It seems likely that a considerable gain could be obtained for late R and Nb spectra by using red sensitive plates.

The spectra of stars of Secchi's fourth type have bands whose heads are on the side of longer wave-lengths. At present these stars find a place in Harvard classes R and N. Some members of another group of stars for which until recently no spectral division had been assigned were also placed in the foregoing classes. These now form spectral class S. Class N includes most of these stars, which had been observed spectroscopically by various investigators,

[1] *Contributions from the Mount Wilson Observatory*, No. 276.

notably, Hale, Ellerman and Parkhurst,[1] Rufus,[2] Shane,[3] and Moore.[4] Merrill,[5] in a preliminary study of the spectra of the stars of class S, has shown that the bands are due to zirconium oxide and that their spectra in general are more closely related to class M than to classes R or N.

The more marked features of the spectra of stars of class R, viz., bands quite certainly to be ascribed to carbon and its compounds, resemble those of class N. In both classes the continuous spectrum is rich in absorption lines for which there are many coincidences. But, in certain regions, especially among early class R stars, there are some marked departures from stars of class N as regards individual lines or relative line intensities, although the differences are not striking on objective-prism plates. R stars, in general, have relatively more light in the blue and violet regions of the spectrum than do stars of class N. This characteristic is most pronounced for early R stars, and decreases as later subdivisions are considered, approaching N stars in this respect in the latest subdivisions. The band absorption goes hand in hand with the foregoing characteristics, in that the bands are weak when the continuous spectrum shows the greatest extension to the violet, and vice versa. It should be borne in mind, however, as Rufus[6] says, that the strength of the most prominent absorption band (head at λ 4736) does not increase progressively through the subdivisions of R to and through N, but that in N-type as in R-type stars there is a sequence from comparatively slight to intense absorption in this region.

The Harvard observers have found between sixty and seventy stars of spectral class R scattered over the sky, of which forty-six are within reach of the 100-inch reflector. Of these 15 per cent are between visual magnitudes 7.0 and 8.0; 24 per cent between 8.0 and 9.0; and 61 per cent are 9.0 or fainter. The color-index varies

[1] *Publications of the Yerkes Observatory*, **2**, 253, 1903.

[2] *Publications of the Observatory, University of Michigan*, **2**, 45, 1916.

[3] *Lick Observatory Bulletins*, **10**, 79, 1918–1923.

[4] *Ibid.*, p. 160.

[5] *Mt. Wilson Contr.*, No. 252; *Astrophysical Journal*, **56**, 457, 1922.

[6] *Publications of the Observatory, University of Michigan*, **2**, 140, 1916.

from early to late R stars, but in the mean is about two magnitudes. Hence a star of visual magnitude 7.0 is not much brighter than 9 photographically. This refers to B.D. magnitudes. Had the magnitudes of the new Henry Draper Catalogue been used, the showing would have been still less favorable. Reference to the notes of Table I will show that two of these stars are variable, but that both fall below the ninth magnitude even at maximum.

Dr. Rufus,[1] of the Observatory of the University of Michigan, has made a very careful study of the spectra of ten of these stars from the standpoint of both chemical constituents and radial velocities. Reference should be made to his paper, especially for his results on constitution. He used a 3-foot reflector with a one-prism spectrograph giving a dispersion of 48 A per mm at λ 4500, and he is to be congratulated upon having secured spectra of ten of these stars with a slit-width of approximately 0.003 inch.

Whatever additional radial velocities could be obtained with the 100-inch reflector without prolonged exposure seemed worth while; and since 1919 an effort has been made to get spectrograms of them whenever the instrument was available and the seeing seemed to warrant it. Unexpected clouds and deterioration in the seeing after starting a long exposure, together with uncertainties in the magnitudes and color-indices and the effects of large zenith distances of stars far to the south, have resulted in considerable unprofitable effort. The spectrograph had a single 60° prism and a camera lens of 18-inch focus with a dispersion of 44 A per mm at λ 4500. The slit-width used was about 0.002 inch. After some experience it was found that an exposure of two hours under good conditions could be expected to yield a spectrogram of sufficient density for a star of visual magnitude 8.0. This criterion and experience with individual cases showed the practicable limit of magnitude for exposures up to five hours. Some stars, because of faintness, great southern declination, or, in one case, because of variability, were photographed with a dispersion of about 97 A per mm at λ 4500, given by replacing the 18-inch camera with one of 7-inch focus. In all, spectra of thirty-one stars were obtained and measured for radial velocity. Ten of them are the stars measured by Rufus,

[1] *Ibid.*, p. 103, 1916.

and serve as a valuable check upon the velocities derived for the remainder.

Naturally, the brighter stars were observed first, and the strongly exposed plates which they gave were measured in considerable detail. With the wave-lengths given by Rufus, velocities were derived which served as a basis for reducing the measured wave-lengths to zero velocity. From the material of nine first-class spectrograms, which were representative of all subdivisions of class R, twenty-seven lines were selected for which mean wave-lengths were obtained. The selection took into consideration both the quality of the line (only those marked good or fair on most of the plates were used) and the range in individual values from the different plates. The latter point was important since one set of wave-lengths was to serve for all divisions of R. It is not suggested that the twenty-seven lines form an exhaustive list. They are the basis of the velocities finally derived by micrometer measurements with a Gaertner comparator from the spectrograms secured with the 18-inch camera.

As a check it seemed advisable to measure the plates in the quite independent manner furnished by the use of the Hartmann spectrocomparator. For this purpose a widened spectrogram of the R star, B.D.+42°2811, Plate C1006, was made for use as a standard. Practically all plates taken with the 18-inch camera have also been measured with the Hartmann machine and this standard.

In the case of the plates of lower dispersion, difficulty was met in getting good measures of lines and bands with the Gaertner comparator, but the use of a suitable standard (again a spectrogram of B.D.+42°2811, Plate C1735) on the Hartmann instrument was found to be quite satisfactory. The manner of deriving the velocities for the two standard plates is explained on page 350.

The usual procedure with the Hartmann spectrocomparator was followed in the case of the plates of higher dispersion, but when measuring plates taken with the 7-inch camera, it was necessary to move the arm carrying the microscopes between settings for coincidence with star lines and with comparison lines; the windows in the reflecting prism are too widely spaced for the length of the comparison spectrum available in this case. The motion, however, is slight and smooth, is normal to the length of the spectrum,

involves no relative disturbance of the two plates compared, and throughout a measurement the order of coincidences (star-lines, comparison-lines) was reversed from section to section. As a matter of fact, measures reproduced themselves in an entirely satisfactory manner for plates of such low dispersion.

Table I contains the data for the individual plates for each star photographed. There are thirty-one stars, twenty-nine of class R, one doubtful, and one of class S. The letter "N" in parentheses following the designation in the first column refers to the notes at the end of the table. The magnitudes are visual, from the Henry Draper Catalogue as far as complete, and beyond that from the B.D. Below the magnitude is given the Harvard spectral classification. The sixth column indicates the number of times each plate has been measured with the Gaertner (left-hand figures) and with the Hartmann comparator. The weights in the seventh column depend primarily upon the quality of the plates and in large measure are relative for the plates of any particular star, although in some cases low weight has been assigned to all the plates, as none of them are of good quality. The eighth column gives the velocity for each plate derived from a mean of its several measures, Gaertner and Hartmann measures being given the same weight, while the ninth column gives the weighted mean radial velocity. For comparison the velocities derived by Rufus are given in the tenth column. The proper motions were computed by R. E. Wilson and kindly sent to me by letter. Plates of low dispersion taken with the 7-inch camera are indicated by an asterisk (*). A few plates whose numbers have a Greek γ appended were obtained by van Maanen with the 60-inch reflector; otherwise the 100-inch telescope was employed. Its series carries the letter *C*.

To ascertain the effect of systematic errors, various comparisons were made, with results as shown in Table II. It is to be expected that the velocities would agree with those of Rufus, provided, of course, we are dealing with stars of constant velocity. In making this comparison the velocity of B.D.$+20^\circ 5071$ derived in notes to Table I, -381.6 km/sec., is taken as Rufus' value. B.D.$+42^\circ 2811$ has been omitted because, through the Hartmann measures, all velocities, in part at least, depend upon it. The mean difference for the remaining nine stars is -0.65 km/sec.

TABLE I

Name	Mag. Sp.	1900 α, δ	λ, β	Plate No.	Gaert. Hart.	Wt.	Velocity, km/sec. Red to ⊙			μ_α	μ_δ
							Individual	Mean	Rufus		
B.D.+53°66	9.7	0^h19^m1	86°, − 8°	C 71	2, 2	1	− 34.6				
H.D.1994	R5	+53°44′		1485	1, 2	½	− 40.8	− 36.3			
B.D.+23°123	8.8	0^h48^m9	91 , −32	C 76	2, 1	1	−226.7				
H.D.5223	R3	+23°32′		211	2, 2	1	234.2				
				770	2, 2	1	245.0				
				1523	1, 1	1	−231.8	−234.4		+0″.148	−0″.018
B.D.−10°513	8.3	2^h30^m2	150 , −58	C 625	2, 1	1	+· 6.0				
H.D.16115	R3	− 9°53′		1529	1, 1	1	14.2				
				1816	1, 1	1	+ 13.2	+ 11.1	+21	+0.003	+0.006
B.D.+57°702	8.1	3^h3^m7	108 , 0	C 64	2, 1	½	− 6.8				
H.D.19557	R5	+57°31′		72	2, 1	1	8.2				
				1530	1, 1	1	4.2				
				γ3022	2, 1	1	− 5.2	− 6.0	−11	−0.001	−0.004
B.D.+61°667	7.9	3^h57^m2	112 , + 8	C 65	2, 1	½	− 9.7				
H.D.25408	R8	+61°32′		212	2, 1	1	14.9				
				1475	2, 1	½	3.3				
				γ2913	2, 1	1	7.2	− 9.5	− 7	−0.018	+0.007
				C1579 (Ha)			(− 14.7)				
B.D.−3°1685	7.5	6^h56^m1	185 , + 2	C 283	2, 1	1	+ 24.7				
H.D.52432	R5	− 3° 6′		1476	2, 1	½	21.1	+ 23.5	+18	−0.014	+0.002
				1580 (Ha)			(19.4)				
B.D.+14°1598	Var	7^h7^m2	171 +12	C 280	2	1	45.2				
H.D.55284 (N)	Rp	+14°46′		288	2	1	35.7				
				904	2	½	39.4				
				1487*	2	½	70.2				
				2146*	2	½	+ 35.6	+ 41.7			

B.D.+24°1686	8.2	7^h25^m8	161, +18	C1615	2, 1	1	+ 35.9				
H.D.59643	R8	+24°44′		2011	1	¼	42.7				
				2039	1	¼	50.1				
				2233*	2	¼	+ 65.2	+ 43.2		−0.004	−0.010
C.D.−29°6735	7.6	8^h42^m4	220, + 9	C 771	2, 1	1	+ 7.4				
H.D.75021	R8	−29°21′		901	2, 1	1	15.3				
				1526	1, 1	1	+ 15.2	+ 12.6		−0.038	−0.011
B.D.+34°1929	9.2	8^h53^m6	157, +42	C 284	2, 1	1	+ 26.1	+ 26.1	+25	+0.009	−0.021
H.D.76846	Ro	+34° 9′									
B.D.+50°1603	9.5	8^h56^m2	135, +43	C 214	1, 2	1	+ 3.9				
H.D.77234	R5	+50°29′		956	2, 2	1	3.0				
				1533	1, 2	1	+ 8.4	+ 5.1		−0.004	−0.002
B.D.+14°2048	8.9	$9^h\ 8^m3$	184, +39	C 289	2, 1	1	+ 1.2	+ 1.2	+ 4	+0.003	−0.012
H.D.79319	R5	+14°37′									
B.D.−13°3407	8.7	11^h30^m7	246, +45	C 285	2, 1	1	+ 6.4				
H.D.100764	Ro	−14° 2′		414	2, 2	1	4.6				
				1569	1, 1	1	+ 3.6	+ 4.9		+0.041	−0.003
B.D.−2°3939	10.6	14^h58^m8	323, +45	C1052*	2, 1	1	− 37.2				
H.D.133332	R5	− 2°28′		1070*	2, 2	1	− 18.9	− 28.0			
C.D.−24°12084	7.4	15^h21^m9	310, +25	C 406	1, 1	¼	+ 63.0				
H.D.137613	Ro	−24°49′		509	2, 1	1	53.7				
				902	2, 1	1	+ 57.8	+ 56.6		+0.004	−0.005
B.D.−14°4371	10.7	$16^h\ 7^m6$	326, +24	C 506*	2, 1	1	+ 25.5				
H.D.145777	Ro	−14°57′		1770*	2	1	+ 10.3	+ 17.9			
B.D.+42°2811	7.7	17^h10^m4	37, +34	C 286	2, 1	1	− 15.0				
H.D.156074 (N)	Ro	+42°15′		355	2, 2	1	6.4				
				1006	2	1	12.0	− 11.1	−25.2		
				1022*	2, 1	½	13.9				
				1735*	1		− 16.2				

TABLE I—*Continued*

Name	Mag. Sp.	1900 α, δ	λ, β	Plate No.	Gaert. Hart.	Wt.	Velocity, km/sec. Red to ⊙			μ_α	μ_δ
							Individual	Mean	Rufus		
B.D.−15°4923	Var	18^h13^m0	344°, − 2°	C 469	2, 2	½	− 18.7				
H.D.168227 (N)	R5	−15°39′		554	2, 1	1	− 19.6	− 19.3		+0″.055	+0″.033
B.D.+ 6°3898	9.1	18^h37^m1	6 , + 4	C 472	1		− 51.3	−·51.3		−0.010	+0.001
H.D.172804 (N)	S	+ 6°43′									
C.D.−31°15954	9.2	18^h40^m0	331 , −14	C1053*	2, 1	1	− 70.6				
H.D.173409	R0	−31°28′		1813*	2	½	− 53.6	− 64.9		+0.01:	±0.00:
C.D.−29°15574	8.9	18^h52^m4	335 , −16	C 507*	2, 1	½	+ 42.6				
H.D.175893	R1	−29°38′		1814*	2	½	+ 42.4	+ 42.5		+0.04:	−0.05:
B.D.−17°5492	10.6	19^h3^m0	347 , −13	C1131*	2, 1	½	− 53.8				
H.D.178316	R0	−17°26′		1817*	2	½	− 35.3	− 44.6			
B.D.−10°5057	7.0	19^h17^m6	354 , −13	C 478	2, 2	1	− 48.0				
H.D.182040	R0	−10°54′		γ2705	1, 1	1	47.2				
				γ4424	1, 1	1	− 52.8	− 49.3	−43.8	+0.004	+0.027
B.D.−0°3883	9.8	19^h52^m5	9 , −16	C 624	2, 2	½	+ 53.5				
H.D. 188934	R3	− 0° 2′		1851	1, 1	½	+ 59.8	+ 56.6			
B.D.−12°5755	9.6	20^h26^m2	1 , −29	C 505*	2, 1	1	− 50.1				
H.D.195435	R5	−12°13′		1771*	2	1	− 53.3	− 51.7			
B.D.+34°4134	9.8	20^h39^m7	44 , − 7	C1132	2, 1	1	+ 20.4				
H.D.197604	R3	+34°43′		1768	1, 1	1	+ 21.4	+ 20.9			
B.D.−19°5030	10.3	20^h43^m2	355 , −36	C1818*	2	1	+ 51.7				
H.D.198140	R0	−19°24′		1916*	2	1	+ 40.7	+ 46.2			

B.D.+20°5071 (N).	8.7 R3	21^h50^m7 +20°34′	47, −27	C 62	2, 2	1	−386.7				
				629	2, 2	1	392.5				
				1769	1, 2	1	364.6				
				1855	1, 1	1	−386.4	−382.6	−49	+0.020	−0.003
B.D.−21°6376	9.0 R0	$23^h\ 6^m3$ −21°33′	11, −68	C 773	1, 1	½	+ 34.6				
				1852	2	½	+ 30.2	+ 32.4			
B.D.+ 5°5223	8.7 R3	23^h44^m0 + 5°50′	66, −53	C 556	2, 2	1	− 21.5	− 21.5	−26.0	+0.013	+0.002
B.D.− 3°5751	9.2 R0	23^h57^m0 − 3°23′	63, −64	C1473	2, 2	1	−141.0				
				1856	1, 2	½	−127.2	−136.4			

NOTES TO TABLE I

H.D.55284. A variable. The velocity has been derived from the emission lines of hydrogen. From the absorption lines on four plates, none of which is well exposed for absorption lines, the mean radial velocity is +52.8. Hence Emiss−Abs = +39.8−52.8 = −13.0 km/sec. Moore found −14 km/sec. for this difference in the case of V Cygni, and it is of the same order as that found by Merrill for Me stars in general. All plates have been secured near maximum light. At minimum the star is about mag. 15. The spectrum bears some resemblance to class S.

H.D.156074. Plate C1006 has been used as the Hartmann standard for one-prism plates taken with the 18-inch camera; plate C1735 for those with the 7-inch camera. The mean velocity is from the first three 18-inch camera plates listed here. The standard velocities for C1006 and C1735 have been derived directly from this mean.

H.D.168227. Variable with very irregular period. No data available for brightness at the time the plates were taken; but probably near maximum. Range 9.0–9.8.

H.D.172804. Reduced with wave-lengths from class R stars. Radial velocity probably only approximate.

B.D.+20°5071. My value totally at variance with Rufus' value but necessary in order to obtain coincidence between spectrograms of this star and others of class R upon the Hartmann spectrocomparator. Using Rufus' reduced wave-lengths for this star making a shift in identification to correspond with the indications above mentioned, and employing Rufus' mean wave-lengths, I find −381.6 km/sec. from a total of nineteen lines. This is in surprisingly good agreement with the value in Table I.

The second comparison gives −0.96 km/sec. as the mean difference for the 18-inch camera plates between measures made with the Gaertner and Hartmann comparator.

The third comparison of the table gives the mean differences, Gaertner *minus* Hartmann, for the plates of lower dispersion. As previously stated, the Gaertner measures are of relatively low weight. The velocities for these plates depend in general more upon the Hartmann than the Gaertner measures.

TABLE II

CHECK UPON SYSTEMATIC ERRORS—RADIAL VELOCITIES OF CLASS R STARS

9 Stars	Sanford−Rufus	= −0.65 km/sec.	
39 Measures	Gaertner−Hartmann	= −0.96 km/sec.	18-in cam. and 1 pr.
6 Measures	Gaertner−Hartmann	= −3.20 km/sec.	7-in cam. and 1 pr.
9 Plates of Class N stars measured against the Hartmann standard for R stars gave	Sanford−Moore	= +2.20 km/sec.	
2 R stars measured against X Cancri gave	X Cancri—Table I	= −4.6 km/sec.	
Venus spectrogram measured against R standard gave	O−C	= 0.0 km/sec.	

The points of similarity between the spectra of classes R and N are such that, with care, a star of the latter type can be measured with the Hartmann against the R-type standard, C1006. Table III gives a list of stars of class N for which there are spectrograms obtained at this observatory with a dispersion similar to that used for class R. Those which bear the Greek letter γ were taken by van Maanen, with the 60-inch telescope; the remainder are by the writer. The velocities have been derived with the Hartmann comparator, using C1006 for a standard. Compared with the values obtained by Moore,[1] they give the mean difference Sanford *minus* Moore = +2.2 km/sec.

The spectra of B.D.+61°667, class R8, of B.D.−3°1685, class R5, and of X Cancri, class Nb, were photographed with plates sensitive to slightly beyond Hα. With X Cancri as a Hartmann standard and

[1] *Lick Observatory Bulletins*, **10**, 163, 1922.

Moore's value of its radial velocity used to derive its standard velocity, the two class-R stars gave

	Against X Cancri	Table I	Δ
B.D.+61° 667	−14.7 km/sec.	− 9.5 km/sec.	−5.2
B.D.− 3 1685	+19.4	+23.5	−4.1
Mean			−4.6

This result has small weight because the number of comparisons is small, the dispersion in the region measured is low, and the quality not of the best. It is interesting to note, however, that the region

TABLE III

CLASS N STARS—RADIAL VELOCITIES

NAME	SP.	α (1900)	δ (1900)	PLATE	VELOCITY, KM/SEC.		Δ (S-M)
					Sanford (Hartmann)	Moore	
B.D.+25°205	Na	1h10m6	+25° 14′	γ2912	+15.7	.....	.....
				4425	+21.1	.....	.....
B.D.+38°1539	Na	6 29.7	+38 31	γ2914	+20.4	+16.0	+4.4
UU Aurigae				C 739	+20.7	+16.0	+4.7
B.D.+17°1973	Nb	8 49.8	+17 37	C 777	+ 4.0	+ 8	−4.0
X Cancri							
U Hydrae	Na	10 32.6	−12 52	γ3343	−18.0	−18.8	+0.8
152 Sch.	Nbp	12 40.4	+45 58	γ3324	+16.9	+11.5	+5.4
B.D.+76°734	Na	19 25.1	+76 23	γ4362	+12.1	+12.0	+0.1
UX Draconis							
B.D.+34°4500	Na	21 37.8	+35 3	γ2555	+13.1	+13.0	+0.1
				4366	+16.9	+13.0	+3.9
19 Piscium	Na	23 41.3	+2 56	γ2556	+17.5	+13.1	+4.4
Mean							+2.2

from λ 4736 to Hα can be used with some advantage for red stars, a matter which will again be referred to.

It is to be noted that Moore[1] from an examination of his measures of class N stars decided that they required systematic corrections. Had these not been applied, the difference Sanford *minus* Moore for

[1] *Ibid.*, p. 162.

N stars would have been increased by the full amount of this correction, or about +5.5. km/sec. Further, the difference for the two R stars, from X Cancri and Table I, would have been a still greater negative value. Both these circumstances are in a sense a further justification of the systematic corrections which Moore applied to his measures, and at the same time connect in a satisfactory manner the radial-velocity systems of N and R stars.

The last comparison of Table II gives the difference between the computed velocity of Venus and that derived by measuring a spectrogram of Venus against C1006. The result is 0.0 km/sec.

The comparisons do not indicate the necessity of a systematic correction of any great magnitude. The evidence from the N stars in Table III and from a comparison of the K-term of the solar-motion solution for R stars with that for N stars indicates that a small negative systematic correction would improve the agreement. None has been applied, however, and the measures appear in Table I as they were derived.

The velocities of the two Hartmann standards C1006 and C1735 depend upon the measures of the first three plates appearing in Table I, the mean for which is −11.1 km/sec. Rufus found −25.2 km/sec. The difference raises a question as to whether we are dealing with a star of constant velocity. The three plates upon which Rufus' velocity depends were all obtained within a period of thirteen days. Unless the period of variation were short it might be expected to show no variation, and he found none. The three plates which furnish the mean velocity −11.1 km/sec. show more range than the three by Rufus, but cannot be said to indicate any variation in velocity over an interval of more than a year. Rufus' plates were taken in March, 1915, the Mount Wilson plates in February and April, 1920, and April, 1921. What the velocity in the intervening years may have been cannot be determined. It might prove profitable to secure a few plates each year in an effort to settle the question. However this may be, the test for systematic errors given in Table II calls for the velocity derived here, or one close to it.

The data are too limited for a satisfactory determination of the solar motion referred to these stars, but the question seemed of

sufficient interest to justify an attempt to derive it from the radial velocities. The following[1] formulae are applicable for such a case:

$$K+x\cos\alpha\cos\delta+y\sin\alpha\cos\delta+z\sin\delta-V=0$$
$$V_0^2=x^2+y^2+z^2$$
$$V_0\cos\alpha_0\cos\delta_0=-x$$
$$V_0\sin\alpha_0\cos\delta_0=-y$$
$$V_0\sin\delta_0\qquad=-z$$

where Campbell's well-known K-term has been introduced. α and δ are the co-ordinates, V the radial velocity of the star, and x, y, and z the equatorial components of the group-motion. V_0, α_0, and δ_0 are, respectively, solar speed, right ascension, and declination of the solar apex; x, y, z, and K are to be determined by a least-squares solution. The material has been used in two ways: (1) including all stars; (2) all stars but the three following whose velocities are greater than 100 km/sec.

B.D.+23°123 ..... −234 km/sec.
+20 5071 ..... −383
− 3 5751 ..... −136

Since all three are negative, the two solutions necessarily give quite different results:

	α_0	δ_0	V_0	K	Arith. Mean Resid. Velocity
(1)	350°	+25°	73 km/sec.	−9.7 km/sec	55 km/sec.
(2)	298	+33	21	+7.9	25

Solution (2), based as it is upon limited data, agrees reasonably well with the values for stars in general, viz.: $\alpha_0=270°$, $\delta_0=+30°$, $V_0=21$ km/sec. for the solar motion. The three stars omitted in this solution give evidence of another kind to be presented later which differentiates them from the remainder of the list. It is of interest that the K-term for this solution is nearly that found by Moore for the N stars, namely, +6.8 km/sec. Had my measures all been reduced by 1 km/sec., the difference Sanford *minus* Moore for N stars would have been +1.2 km/sec., the difference for the

[1] Campbell, *Stellar Motions*, p. 172, 1913.

two class R stars measured against X Cancri and as given in Table I would have been -3.6 km/sec., and the K-terms for R and N stars would have been closely alike.

As far as I am aware, there are no measured parallaxes for any of these stars, but R. E. Wilson has derived proper motions for eighteen of them. It should be borne in mind, however, that the material with which he had to work was both scanty and lacking in the precision necessary for the accurate determination of proper motions. Hence any conclusions based upon them must be accepted with some reserve. But since the proper motions are all very small, and the stars for the most part faint, we may conclude that even present-day refinements in parallax determinations by trigonometric methods will be comparatively barren of positive results. Any clue as to luminosity afforded by the data will therefore be welcome, and perhaps through comparison with the luminosities of closely related spectral classes we may judge of the reasonableness of the result.

Although both proper motions and radial velocities are available for but eighteen stars, an attempt has been made to obtain an approximation for the parallax from both parallactic and peculiar motions. Since the first two of the three stars excluded from the second solution for the solar motion are among those for which proper motions are known, the calculation was repeated for the remaining sixteen stars. In obtaining the parallactic and peculiar motions, $a_0 = 270°$, $\delta_0 = +30°$, were used for the solar apex, but the solar speed and mean residual velocities used were those derived in this paper. The four approximations for the mean parallax thus found are:

	18 Stars	16 Stars
Parallactic motion	+0".0006	+0".0011
Peculiar motion	+0.0003	+0.0022
Mean of all	+0".00105	.....

The adopted result, +0".00105, is the simple mean of the four values, and the equivalent of giving half-weight to the two stars of large velocity. The mean apparent magnitude of the stars here

considered is 8.4, which, by the formula $M = m + 5 + 5 \log \pi$, leads to the approximate mean absolute magnitude $M = -1.5$. This is identical with the value derived for N stars by Moore.[1] Other values for stars of class N are -1.3 by Luplau-Janssen and Haarh,[2] and -2.6 by Kapteyn.[3]

Rufus has assembled various criteria which establish the spectral sequence in the order G, R, N. This suggests an examination of the line-pairs for absolute magnitude to determine whether they are sensitive in spectral class R as well as in G. The one pair available is $\lambda\lambda$ 4454 and 4462. The relative intensity $\frac{\lambda\ 4454}{\lambda\ 4462}$ has been estimated for those stars which have proper motions. The results have been combined in four different ways: (1) line intensity against reduced proper motion; (2) line intensity against $100\,\mu$; (3) line intensity against spectral type; and (4) line intensity against radial velocity regardless of sign. (3) shows practically no change, whence we may have some confidence in the run of the other three comparisons as displayed by all subdivisions. (1) and (2) show that λ 4454 becomes stronger with increasing proper motion, and (4) displays an increase in λ 4454 with increase in radial velocity regardless of sign. Both tendencies agree with the behavior of this line-pair in stars of class G. To go farther would be idle speculation, for we possess no parallaxes, and the proper motions are few and relatively inaccurate.

It would not be safe to use the mean absolute magnitude for individual stars in an attempt to combine the radial velocities and proper motions into space motions. It may be of interest, however, to state that such a procedure results in space motions whose projections, upon the plane of the galaxy, display the same asymmetry which Strömberg[4] has studied in several classes of celestial objects. B.D.$+3^\circ 123$ and B.D.$+50^\circ 2071$ agree in a striking way in this respect with the direction of motion for other stars and objects of high speed. With the assumption of any absolute magnitude

[1] *Lick Observatory Bulletins*, **10**, 168, 1922.

[2] *Astronomische Nachrichten*, **214**, 383, 1921.

[3] *Astrophysical Journal*, **32**, 91, 1910.

[4] *Mt. Wilson Contr.*, No. 275; *Astrophysical Journal*, **59**, 228, 1924.

between -1.5 and $+6$ these two stars have apices falling within the quadrant 90° to 270° of galactic longitude. In fact, the apex of B.D.$+20°5071$ scarcely changes for the foregoing range in absolute magnitude. Because these two stars stand out so decidedly from the rest as regards space motion, it seems justifiable to base

TABLE IV

Name	Plate No	1900		Sp. Class	Mag.	Velocity
		α	δ			
B.D.$+42°2811$	$\Sigma 83$	$17^h 10^m.4$	$+42° 15'$	Ro	7.3	.....
C.D.-26 12879	87	18 4.1	-26 40	R3	10.0	+ 10km/sec.
B.D.$+85$ 332	84	19 44.0	$+85$ 9	R3	9.2	+ 20
B.D.$+20$ 5071	81	21 59.7	$+20$ 34	R3	8.7	-420
(Not in B.D.).....	88	22 49.5	$-$ 7 30	Ro	10.5±	$-$ 30
(Not in B.D.).....	85	23 6.2	$+45$ 46	Ro	10.7±	$-$ 10

NOTES TO TABLE IV

B.D.$+42°2811$. This spectrogram was deliberately overexposed in order to bring out the continuous spectrum in the violet and the ultra-violet. Besides an overexposed continuous spectrum to λ 3900, giving strong H and K of calcium, it shows cyanogen group IV, with head at λ 3884, as a very conspicuous feature. The continuous spectrum thereafter increases in intensity, reaching a maximum presumably at the head of cyanogen group V, λ 3590. Five lines or blends, which may be groups of the stronger lines of the cyanogen spectrum, are easily seen within the region of continuous spectrum between the heads of groups IV and V.

B.D.$-26°12879$. The spectrum is strong enough to show that the band of cyanogen group III is very conspicuous.

B.D.$+85°332$. The declination puts this star beyond the reach of the 100-inch reflector. This spectrogram, obtained with the 60-inch telescope, is quite similar to B.D.$-26°12879$.

B.D.$+20°5071$. The velocity here secured is evidence of the reality of the large value, -382.6 km/sec., given in Table I.

The fifth star is not found in B.D. The heads of cyanogen bands III and IV are well shown.

The sixth star is not in B.D. Cyanogen III and IV are present, and a trace of continuous spectrum to the violet of the latter.

the solar motion and mean absolute magnitude chiefly on the remaining data.

Some interest attaches to six spectrograms (Table IV) taken with a focal-plane spectrograph whose mean dispersion between H and K of calcium and Hβ is about 200 A per mm. The last column of the table gives results derived from the H and K lines

for four stars whose velocities were not otherwise determined, and for B.D.+20°5071. The last checks the large velocity appearing in Table I. The velocities have been rounded off to even tens, and are to be considered as showing only their order of size. The magnitudes for the last two stars have been estimated from the relative exposure times of the spectrograms, and are only crude approximations.

Reference has been made on page 348 to spectrograms of R and N stars taken with plates sensitive to a region extending slightly beyond Hα. The plates used were Ilford Special Rapid Panchromatic ammoniated. A comparison of these spectrograms with others on Seed's Gilt Edge 30, reduced to the same seeing, shows that the exposure times for the former to those for the latter are:

R5	50 per cent
R8	36 per cent
Nb	10 per cent

For both kinds of plates the exposure is so chosen that some of the weaker portions of the spectrum toward the blue can be used without overexposing the strongest part recorded. In stars of spectral class R5 to R8 it would be possible to make use of the spectrum from λ 4736 to a point slightly beyond Hα (mean dispersion 98 A per mm). In the Nb spectrum (X Cancri) the region that could be used extends to the same point in the red, but toward the violet terminates around λ 5200, giving a mean dispersion of 110 A per mm. In other stars of class N, the extension to the violet may be considerably less. Dispersion of this order should give measures of velocity comparable in precision with those from spectrograms taken with the 7-inch camera and Seed's 30 plates. The use of two prisms for the foregoing regions would provide a dispersion comparable with that of the one-prism spectrograph and the 18-inch camera for the region covered by Seed's 30 plates. In this case, exposures with ammoniated Ilford Special Rapid Panchromatic plates would be no longer for a star of spectral class R5, and probably would be shorter for R8- and N-type stars.

MOUNT WILSON OBSERVATORY
December 1923

THE ORBITS OF TWO DWARF SPECTROSCOPIC BINARIES[1]

By R. F. SANFORD

ABSTRACT

Orbits of two dwarf spectroscopic binaries.—The elements of H.D.27130 and H.D.96511, as determined from more than twenty spectrograms in each case, are, respectively: P, 5.6100 and 18.8922 days; e, 0.040 and 0.282; K, 58.6 and 40.0 km/sec.; ω, 88°.6 and 332°.97; T, J.D.2423102.193 and 2423154.071; γ, +38.2 and −46.8 km/sec.; $a \sin i$, 4,516,900 and 9,975,000 km; $\frac{m_1^3 \sin^3 i}{(m+m_1)^2}$, 0.1166 and 0.1111 $\odot$

The membership of H.D.27130 in the Taurus stream is definitely established, for its speed, 42.6 km/sec. toward $\alpha = 80°.3$ and $\delta = +7°.2$, agrees as well as could be expected with the corresponding quantities for the stream as a whole as determined by Boss; viz., 45.6 km/sec., $\alpha = 91°.8$, $\delta = +6°.9$.

In continuation of three contributions,[2] already published, relating to the orbits of spectroscopic binaries, the present paper gives results for the orbits of the two stars listed in Table I. The general remarks of *Contribution* No. 201 apply here as well. The details for each star follow.

TABLE I

	Visual Mag.	α 1900	δ 1900	Sp.	Visual Mag.	μ	π Sp.	No. Rev.
H.D.27130..	8.3	$4^h\ 11^m.9$	+16° 42′	G7	5.5	0″.110	0″.028	256
H.D.96511..	7.1	11 2.2	+82 17	G0	4.4±	0.230	0.029±	44

H.D.27130[3]

This star, No. 12 in *Groningen Publication* No. 14, is there given as a probable member of the Taurus group. Since stars which certainly belong to this group are characterized by a very definite velocity, H.D.27130 was placed on the Mount Wilson program both to find whether its radial velocity conformed to that of the group and to obtain its absolute magnitude. Its apparent magni-

[1] *Contributions from the Mount Wilson Observatory*, No. 277.

[2] *Mt. Wilson Contr.*, Nos. 201, 221, 251; *Astrophysical Journal*, **53**, 201, 1921; **55**, 30; **56**, 446, 1922.

[3] B.D.+16°577; A. G. Berlin, A 1126; Cin. 570.

tude places it beyond reasonable exposure-times except with the 100-inch telescope, and for all plates but one this telescope has been used, with the one-prism spectrograph referred to in previous papers. The velocity from the second plate differed so much from that from the first as to leave little doubt that this star is a spectroscopic binary. The spectrum, G7 with dwarf characteristics, is at all times single and refers to the primary of the binary.

Before many plates had been taken it became apparent that 5.6 days is a good approximation to the period, and that the orbit differs little if any from a circle. With these two features as a guide, favorable times of observation were chosen, as far as possible, until twenty-two plates had been obtained, when it was decided that the elements could be determined with requisite accuracy.

The interval between the first and last observations (256 orbital revolutions) is sufficient to establish that $P = 5.6100$ days is correct within 0.002 day; this element has therefore been adopted as final and used to assemble all the observations at one epoch. A free-hand curve through the plot was the basis of the other five preliminary elements derived by Russell's method. These were corrected by the method of least-squares, using Schlesinger's notation. Inadvertently Plate C792 was omitted in forming the conditional equations so that the normal equations are based upon twenty-one plates. The weights depend upon quality both in density and focus. The omitted plate, C792, is given in Table II together with another plate, C2456, obtained after a start had been made in correcting the elements.

Table III contains the preliminary elements, their corrections, the final elements, probable errors, and the values of $a \sin i$ and the mass function. The probable errors of the elements are based upon a probable error of ± 3.1 km/sec. in the radial velocity from a plate of unit weight, which appears in the radial-velocity diagram, Figure 1, as the radius of the circles representing the individual observations of velocity.

Adams and his collaborators[1] give the spectroscopic parallax of this star as $+0''.028$. The velocity of the system, γ, is here found to be $+38.2$ km/sec. When these quantities are combined with

[1] *Mt. Wilson Contr.*, No. 199, p. 31, 1921; *Astrophysical Journal*, **53**, 43, 1921.

the proper motion, it is found that the star is traveling with a speed of 42.6 km/sec. toward $\alpha = 89°.3$ and $\delta = +7°.2$. Boss's value for

TABLE II

OBSERVATIONS OF H.D.27130

Plate No.	Date	G.M.T.	Phase	Velocity	O−C	Wt.
				km/sec.	km/sec.	
C 148	1919 Oct. 18	21h 09m	1d.407	−21.6	−1.3	1.0
290	1920 Mar. 2	15 55	2.550	+18.8	2.5	0.5
634	Sept. 1	23 19	0.729	− 7.0	1.5	1.0
659	Sept. 20	23 27	2.905	+38.0	4.5	0.7
792	Nov. 27	20 21	3.454	73.6	0.2	.....
807	Dec. 21	16 42	4.803	79.0	−6.2	0.5
862	1921 Jan. 24	20 12	5.349	+58.5	+0.2	0.5
1509	1922 Jan. 8	20 29	0.931	− 3.6	+9.7	0.5
1524	Jan. 13	18 30	0.239	+22.8	−0.6	1.0
1531	Jan. 14	19 32	1.272	−28.0	−8.1	1.0
1566	Feb. 15	17 48	5.159	+79.4	+9.1	0.5
1572	Feb. 16	15 21	0.447	+ 8.7	−1.2	1.0
1578	Feb. 17	15 09	1.438	−20.4	−0.4	1.0
1595	Mar. 8	16 08	3.647	+85.2	+3.0	0.5
1629	Mar. 20	15 50	4.416	94.7	−1.6	0.5
1918	Oct. 3	23 43	5.394	+54.9	−0.4	1.0
1925	Oct. 6	0 22	1.812	− 4.4	+8.8	1.0
1938	Oct. 8	0 10	3.804	+92.3	+4.3	1.0
2038	1923 Jan. 1	20 35	5.504	+44.2	−3.6	1.0
γ11501	Jan. 3	18 47	1.819	−18.8	−5.9	0.5
C 2076	Jan. 25	17 17	1.317	−18.4	+1.8	1.0
2134	Feb. 25	17 02	4.257	+94.2	−2.5	0.5
2456	Sept. 26	22 27	4.302	+92.1	−4.8	.....

TABLE III

ELEMENTS OF H.D.27130

	Prelim. Elements	Corrections	Final Elements
P	5.6100	.....	5.6100 days
e	0.09	−0.050	0.040±0.006
ω	98°.1	9.5	88.6±4°.3
K	59.5	0.90	58.6±0.36 km/sec.
T	J.D.2423102.300	0.107	J.D.2423102.193±0.063
γ	+39.2	−1.05	+38.2 km/sec.
$a \sin i$	.....	.....	4,516,900 km
$\frac{m_1^3 \sin^3 i}{(m+m_1)^2}$	.....	.....	0.1166 ⊙

the Taurus stream is +45.6 km/sec. toward $\alpha = 91°.8$ and $\delta = +6°.9$. The agreement of the space-velocity of this star with that for the

group is ample to remove any doubt concerning its membership in the group.

H.D.96511[1]

The spectrum of this star is G0 and has the absolute-magnitude lines of a dwarf. A difference of nearly 70 km/sec. between the measures of the second and third spectrograms established the variability of its radial velocity and its probable binary character. Upon none of the plates is there any certain evidence of the spectrum of the secondary.

The period of velocity variation was found, after comparatively few plates had been taken, to be close to 18.89 days, and was used

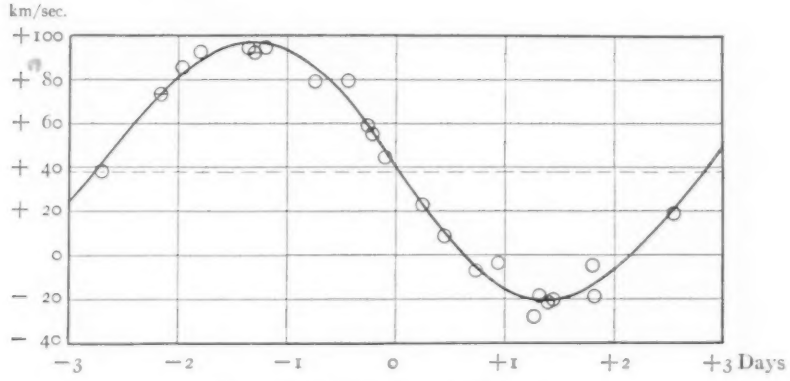

FIG. 1.—Velocity-curve of H.D.27130

for the advantageous choice of epochs for the other plates. The twenty-five spectrograms listed in Table IV define the velocity-curve with sufficient accuracy for a satisfactory determination of the elements. The one-prism spectrograph attached to the 60-inch reflector, already referred to repeatedly in previous papers, was used for all the plates. The declination of H.D.96511 puts it beyond the reach of the 100-inch reflector.

When assembled with a period of 18.89 days the twenty-five observations gave by Russell's method a set of preliminary elements all of which it seemed advisable to correct by a least-squares solution. The period was included because the total interval between the

[1] B.D.+82°325; Lalande F. 1800.

first and last observations involves only about 44 orbital revolutions. The differences between the residuals from ephemeris and conditional equations were rather large after the elements had been thus

TABLE IV

OBSERVATIONS OF H.D.96511

Plate No.	Date	G.M.T.	Phase	Velocity	O−C	Wt.
				km/sec.	km/sec.	
γ 9951	1921 Feb. 24	$23^h\ 40^m$	$7^d.544$	−62.4	+0.8	1.0
10001	Mar. 16	23 47	8.644	70.3	−1.1	1.0
10183	June 11	18 42	0.983	1.8	4.8	1.0
10735	1922 Jan. 8	21 29	4.285	37.9	−3.5	0.5
10753	Jan. 12	20 20	8.237	66.4	+0.8	0.5
10769	Jan. 15	20 08	11.228	80.0	−3.6	1.0
10827	Feb. 15	18 57	4.396	28.5	+7.1	0.5
10849	Feb. 18	18 18	7.369	63.6	−1.4	1.0
10892	Mar. 9	23 26	7.690	60.2	+3.8	0.5
10920	Mar. 17	22 45	15.661	52.6	+4.6	1.0
11001	Apr. 18	21 02	9.806	74.8	−1.2	0.5
11039	May 11	18 36	13.812	70.9	+1.5	0.5
11092	June 7	16 28	2.939	19.2	−2.2	1.0
11098	June 8	16 22	3.935	28.3	+1.7	1.0
11119	July 2	17 10	9.075	71.0	0.0	1.0
11125	July 3	16 18	10.039	74.1	+0.1	1.0
11131	July 4	17 10	11.075	76.2	0.0	0.5
11161	July 10	16 40	17.055	−41.2	−6.4	1.0
11173	July 12	16 20	0.149	+ 3.4	+3.0	1.0
11180	July 13	16 23	1.151	+ 6.5	+4.1	1.0
11698	1923 Apr. 3	15 58	0.644	− 0.5	−3.7	1.0
11762	May 5	19 35	13.900	67.1	+4.9	1.0
11768	May 6	19 05	14.880	69.8	−4.4	1.0
11810	May 28	17 56	17.941	−14.4	+2.4	1.0
11816	May 29	18 04	0.054	+ 2.2	+2.4	1.0

TABLE V

ELEMENTS OF H.D.96511

	Preliminary Elements	Corrections		Final Elements
		First Solution	Second Solution	
P	18.89	+0.00217	.....	18.8922±0.0061 days
e	0.30	−0.0257	+0.0075	0.282±0.017
ω	316°.00	+15°.85	1°.12	332.97±4°.23
K	41.2	−1.52	0.33	40.01±0.66 km/sec.
T	J.D.2423153.315	+0.710	0.046	J.D.2423154.071±0.178
γ	−45.9	−0.92	+0.07	−46.75 km/sec.
$a \sin^3 i$	.....	.....	.....	9,975,000 km
$\frac{m_1^3 \sin^3 i}{(m+m_1)^2}$	.....	.....	.....	0.1111 ⊙

corrected, and a second correction seemed desirable. For this the period was considered as known. Table V gives the preliminary elements, the two sets of corrections, and the final elements and their probable errors, based on ±2.3 km/sec. as the probable error of a velocity from a single plate of unit weight. The agreement between the velocity-curve based upon the final elements and the

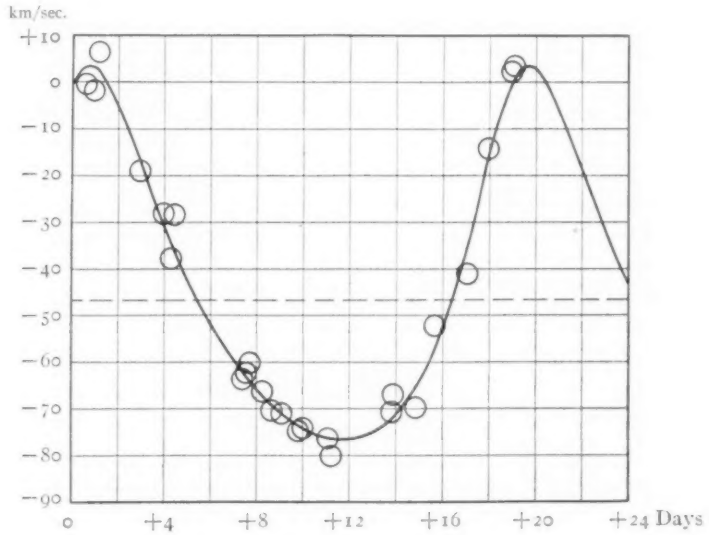

FIG. 2.—Velocity-curve of H.D.96511

twenty-five observed velocities is shown graphically in the velocity-curve (Fig. 2).

The bearing which the elements of these two dwarf binaries may have upon the relation between period and eccentricity in giant and dwarf spectroscopic binaries has been anticipated in the concluding paragraphs of *Contribution* No. 251, and need not be repeated here.

MOUNT WILSON OBSERVATORY
February 1924

Reviews

The Principle of Relativity with Applications to Physical Science. By A. N. Whitehead. Cambridge: University Press, 1922. Pp. 190.

This "alternative rendering of the theory of relativity" is arranged conveniently in three parts: first, an exposition of features of the author's personal philosophy of physical science; second, the development of formulae connected with physical applications; and, third, a purely mathematical sketch of the theory of tensors.

The theory expounded is in effect mainly equivalent to Einstein's, though avowedly different in philosophic cast. In point of view it agrees in emphasizing the fundamental character of the concept of natural time and the relativity of stratification by co-ordinate systems, but differs in insisting on a distinction between physics and geometry. The latter is understood as some scheme of "uniform relatedness," serving as a frame for the tabulation of physical events. For development of details this frame is, in what seems a rather arbitrary way, taken in the form of the four-dimensional geometry of the special relativity, though the physical theory has the wider scope of Einstein's more general theory.

The topics of chief astronomical interest relate to certain residual effects, such as are indicated by Einstein's theories, but in connection with which Whitehead's basis of argument naturally suggests assumptions leading to somewhat different numerical values. The differences are mostly too small for present detection, but the results contain some suggestions of interest; for example, concerning the interpretation of solar optics.

A. C. L.

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